JOURNAL OF ALGEBRA **200,** 650–700 (1998)
ARTICLE NO. JA977231

Toral Rank One Simple Lie Algebras of Low Characteristics

Serge Skryabin

Chebotarev Research Institute of Mathematics and Mechanics, Kazan, Russian and Mathematisches Seminar der Universität Hamburg, Bundesstr. 55, 20146, Hamburg, Germany

Communicated by Georgia Benkart

Received October 12, 1995

INTRODUCTION

Recent years have seen a successful completion of the classification of simple finite dimensional Lie algebras over algebraically closed fields of characteristic $p > 7$. The work on characteristics $5, 7$ is in progress now. Much more difficult are cases of characteristics $2, 3$ which display an abundance of simple Lie algebras that fit into neither of the two major classes: classical and Cartan type algebras (see [19] where most of the known examples in characteristic 3 are discussed). Only a few classification results under very restrictive hypotheses have been known.

The general classification scheme developed by R. E. Block and R. L. Wilson in case of restricted Lie algebras [3] and then extended by H. Strade to the nonrestricted case [23] is based on the study of root space decompositions $\sum L_\alpha$ of a simple Lie algebra L with respect to tori T of maximal dimension in the minimal p-envelope of L. The dimension of those tori is called the (absolute) toral rank of L [22]. With each subgroup $\Gamma \subset T^*$ generated by r roots linearly independent over the prime field one associates a rank r section $\sum_{\alpha \in \Gamma} L_\alpha \subset L$ and tries to glue together information about sections of low rank and their cores, the factor algebras by solvable radicals. As a first step, one has to determine the cores of rank one sections. That leads to the necessity to know toral rank one simple Lie algebras. In the present paper we solve this problem for $p = 2, 3$.

The main result, Theorem 6.5, states that only simple Lie algebras sl(2) and psl(3), i.e., the factor algebra of sl(3) by its one-dimensional center,

0021-8693/98 $25.00

have toral rank one in characteristic 3. Both of them are classical. No simple Lie algebras of toral rank one exist in characteristic 2.

This result exposes a drastic difference pertaining to the whole classification process in characteristics 2, 3: rank one sections do not distinguish classical Lie algebras. Recall a result of H. Strade saying that for $p > 7$, a simple Lie algebra is classical if and only if every one of its rank one sections with respect to an optimal torus is either solvable or has core sl(2) and there exists at least one nonsolvable rank one section [24, Theorem 4.5]. Our list of toral rank one simple Lie algebras is even shorter than that for larger characteristics $(\mathrm{sl}(2), W_1(1), H_2''(\mathbf{1}))$. If $p = 3$ then Cartan subalgebras of those Lie algebras are one-dimensional, and therefore we get precisely the same two algebras that were found by I. Kaplansky in his determination of simple Lie algebras with a one-dimensional Cartan subalgebra and all roots in the prime field [8]. It can be anticipated that each step of the classification for larger characteristics will require for $p = 2, 3$ its counterpart in higher ranks.

The work on toral rank one actually produces a more general result which is important for the classification in its own right: the determination of simple Lie algebras L with a toral rank one Cartan subalgebra H. The toral rank of H in L is, by definition, the maximum of dimensions of tori in the restricted subalgebra generated by H in the minimal p-envelope of L. It is in these settings that the problem was solved by R. L. Wilson for $p > 7$ [32]. A paper of G. Benkart and J. M. Osborn [1] contains a more detailed description of toral rank one Cartan subalgebras in simple Lie algebras. Later, A. A. Premet generalized Wilson's result to characteristics 5, 7 [15]. Here we investigate cases $p = 2, 3$ (Theorem 6.3). Suppose that L_0 is a maximal subalgebra of L containing H. If $\dim L/L_0 = 1$ or $\dim L/L_0 = 2$ then L is a Lie algebra of Cartan type, essentially the same as those occurring in larger characteristics. For $p = 3$ there are isomorphisms $W_1(1) \cong \mathrm{sl}(2)$ and $H_2''(\mathbf{1}) \cong \mathrm{psl}(\mathbf{3})$. There is only one possibility more: when $p = 2$ and $\dim L/L_0 = 4$. In this case we determine the associated graded algebras and give examples of L. No attempt is made to obtain their complete classification or to identify them with the known simple algebras. To show that the absolute toral rank of these algebras is greater than 1 we need Theorem 5.1 which states that the toral rank of a graded Lie algebra can never exceed the toral rank of its filtered deformation.

The classification of toral rank one simple Lie algebras in big characteristics rests heavily on Wilson's triangulability theorem which claims that for $p > 7$ every Cartan subalgebra H in a simple Lie algebra L is triangulable, i.e., $[H, H]$ acts in L nilpotently [31]. A generalization to $p = 5, 7$ was given by Premet [15]. Already for $p = 5$ there exist simple algebras with nontriangulable Cartan subalgebras. Still many more examples are found in characteristic 3. We shall see that toral rank one Cartan subalgebras are nevertheless triangulable even for $p = 3$. However, it seems to be hardly possible to prove this fact using a modification of Wilson's arguments. A. A. Premet informed me that he attempted to do it, but was able only to obtain an estimate 18 for the codimension of maximal subalgebras containing a Cartan subalgebra. If $p = 2$ then triangulability breaks down even for toral rank one Cartan subalgebras. Consider, for example, a simple hamiltonian algebra $L = H_2''(\mathbf{m})$ with its standard filtration $L = L_{-1} \supset L_0 \supset \dots$. If H is a Cartan subalgebra of L contained in L_0 then H has toral rank 1 in L. The image $\overline{H}$ of H in $L_0/L_1 \cong \mathrm{sl}(2)$ is a Cartan subalgebra of the latter. On the other hand, sl(2) is nilpotent when $p = 2$; in fact it is isomorphic to the three-dimensional Heisenberg algebra. Hence $\overline{H} = L_0/L_1$, and the action of this algebra in its natural two-dimensional module L/L_0 is not triangulable.

For these reasons we are limited to direct application of the filtration method. As usual, we have to solve several questions. The first one is to determine the associated graded algebra. We encounter graded algebras G with a nilpotent component G_0 or, in one case in characteristic 3, G_0 has its own gradation with a nilpotent null component. Next we have to show that certain graded semisimple Lie algebras whose minimal ideal has a nontrivial centroid cannot be associated with filtrations of simple Lie algebras or, if they do occur in characteristic 2, there is a limited number of possibilities. Even knowing that G is a Lie algebra of Cartan type, we still have to work to show that its filtered deformation L is itself of Cartan type. There is one difficult case in characteristic 3 when only having done a subtle job of changing the filtration we were able to prove that L is hamiltonian. The source of this difficulty lies in the fact that there is a certain class of exceptional hamiltonian algebras which contain an infinite family of maximal subalgebras of codimension 2. We study these algebras more thoroughly in the final section of the paper. A large part of our work presents an interest even for big characteristics and may find further applications. The proof of final results, however, rests on special properties of filtration peculiar to low characteristics.

Throughout the paper the ground field k is assumed to be algebraically closed of characteristic $p > 0$. The term "algebra" will normally mean a finite dimensional algebra over k. In a few places we do encounter infinite dimensional algebras or algebras over a commutative ring. Our notations of distinguished ideals in Cartan type Lie algebras follow [18]. I chose to denote by $K_1'(m)$ the commutant of the Zassenhaus algebra $W_1(m)$ when $p = 2$. It has as a basis the elements $x^{(r)}\partial, 0 \le r < 2^m - 1$. Although the contact algebras $K_n(\mathbf{m}, \omega)$ are usually considered for $n \ge 3$, their definition makes sense for $n = 1$ with $K_1(m, \omega) = W_1(m)$ and the property that

$K_{2s-1}(\mathbf{m}, \omega)$ has a commutant of codimension 1 exactly when $s + 1 \equiv 0 \pmod p$ remains in force for $s = 1$.

1. GRADED LIE ALGEBRAS WITH A NILPOTENT NULL COMPONENT

For a $\mathbb{Z}$-graded Lie algebra $G = \bigoplus_{i \in \mathbb{Z}} G_i$ we put $G_+ = \bigoplus_{i > 0} G_i$ and $G_- = \bigoplus_{i < 0} G_i$. Of particular importance in the classification of simple Lie algebras are conditions listed below:

(g1) $C_G(G_-) \cap (G_0 + G_+) = 0$, i.e., $G_0 + G_+$ contains no nonzero ideals of G,

(g2) G_- is generated by G_{-1},

(g3) G_{-1} is an irreducible G_0-module,

(g4) $C_G(G_+) \cap G_- = 0$, i.e., G_- contains no nonzero ideals of G,

where C_G stands for the centralizer of a subspace in G. Denote by $A(G)$ the homogeneous ideal of G generated by G_{-1} and by $M(G)$ the maximal homogeneous ideal of G contained in G_-. If G satisfies (g1)–(g3) and $G_1 \neq 0$ then $M(G) \cap G_{-1} = 0$, i.e., $M(G) \subset \bigoplus_{i < -1} G_i$. The factor algebra $G/M(G)$ satisfies then (g1)–(g4). In this section we aim at the following result:

THEOREM 1.1. *Let G be a graded Lie algebra satisfying* (g1)–(g3). *Assume that G_0 is nilpotent and $[G_{-1}, G_1] = G_0$. Then one of the following three cases occurs*:

(1) $p > 2$, $\dim G_{-1} = 1$, $G_{-2} = 0$, $G \cong \mathrm{sl}(2)$ *or* $G \cong W_1(m)$;

(2) $p = 2$, $\dim G_{-1} = 1$, $G_{-2} = 0$, $G \cong W_1(m)$ *or* $G \cong K_1'(m) := [W_1(m), W_1(m)]$;

(3) $p = 2$, $\dim G_{-1} = 2$, $G_0 \cong \mathrm{sl}(G_{-1})$, $[G_{-2}, G_1] = 0$. *If in addition* $G_{-2} = 0$ *then G is of hamiltonian type*, $H_2''(m_1, m_2) \subset G \subset H_2(m_1, m_2)$, $m_1, m_2 > 1$.

In a series of papers M. I. Kuznetsov intensively studied graded Lie algebras of depth 1, i.e., satisfying $G_i = 0$ for $i < -1$. In particular, the case of algebras of depth 1 and characteristic $p > 2$ in Theorem 1.1 is covered by [11, 12]. In the course of classification of simple Lie algebras with a solvable maximal subalgebra B. Weisfeiler dealt with graded Lie algebras having a solvable null component and an arbitrary depth [29]. He could determine their structure only for $p > 5$, however. Special properties of nilpotent algebras enable us to give a significantly simpler proof of

Theorem 1.1 which involves no restrictions on depth or characteristic. Our main tool is the following fundamental result due to Weisfeiler [28]:

THEOREM 1.2. *Let G be a graded Lie algebra satisfying* (g1)–(g4). *Then G is semisimple and $A(G)$ is its unique minimal ideal*, $[A(G), A(G)] = A(G)$. *Suppose that* $A(G)_+ := A(G) \cap G_+ \neq 0$. *Then*

(1) *$A(G)$ is G_0-simple*;

(2) *There exists a simple graded Lie algebra S and a commutative associative algebra B such that $A(G) \cong B \otimes S$, $A(G)_i \cong B \otimes S_i$ for every $i \in \mathbb{Z}$*;

(3) *S satisfies* (g1), (g2), (g4).

A short proof of this theorem based on the interpretation of derivation simple objects as coinduced modules was given by H. Strade [25]. The theorem is a special version for graded algebras of results of R. Block [2]. Note, in particular, that B is identified with the centroid of $A(G)$, i.e., with the algebra of those linear endomorphisms of $A(G)$ that commute with all its adjoint derivations. Thus, under assumption $A(G)_+ \neq 0$, all homogeneous components of $A(G)$ are stable under the action of B. Furthermore, G acts on B as a Lie algebra of derivations, and item (1) of the theorem means precisely that B is G_0-simple, i.e., B contains no nonzero proper G_0-invariant ideals. If $i \neq 0$ then the action of G_i on B produces endomorphisms of $A(G)$ which shift the degrees of gradation. Since B contains no such endomorphisms, we see that both G_- and G_+ annihilate B. If G_0 is solvable, then so is $G_0 + G_-$ as well, hence the condition $A(G)_+ \neq 0$ is fulfilled automatically.

LEMMA 1.3. *Let G be a graded Lie algebra satisfying* (g1)–(g3). *Suppose that $A(G)_+ \neq 0$. Let I be a homogeneous ideal of $A(G)$. Then $I = A(G)$ whenever $I \supset G_{-1}$ or $I \supset A(G)_1$, and $I \subset M(G)$ whenever $I \cap G_{-1} = 0$. Furthermore*, $[A(G), A(G)] = A(G)$.

Proof. If $A(G) = I + M(G)$, then $I \supset G_{-1}$ because $M(G) \cap G_{-1} = 0$, whence $I \supset G_-$ by (g2), and it follows $I = A(G)$. So it will suffice to give the proof replacing G with $G/M(G)$ and I with its image in $G/M(G)$. In particular, this applies to $I = [A(G), A(G)]$. We may assume therefore that G satisfies (g4) as well. We may identify the algebra S in the decomposition $A(G) \cong B \otimes S$ of Theorem 1.2 with a subalgebra of $A(G)$. The adjoint representation of S in $A(G)$ is completely reducible with irreducible subrepresentations all isomorphic to the adjoint representation of S in itself. Moreover, any irreducible S-submodule in $A(G)$ is of the form $f \otimes S$ for some $f \in B$. Now I, being S-invariant, is the sum of irreducible S-submodules contained in I. Hence $I = J \otimes S$ for a certain subspace $J \subset B$. If $I \supset G_{-1}$ or $I \supset A(G)_1$ then $J \otimes S_{-1} = B \otimes S_{-1}$ or

$J \otimes S_1 = B \otimes S_1$, respectively, whence $J = B$, and it follows $I = A(G)$. If $I \cap G_{-1} = 0$ then $J \otimes S_{-1} = 0$, whence $J = 0$, and it follows $I = 0$.

Recall that every finite dimensional representation of a nilpotent Lie algebra is a direct sum of subrepresentations corresponding to different eigenvalue functions. In particular, with each irreducible representation there is associated its eigenvalue function. Suppose that G is a graded Lie algebra satisfying (g1)–(g3) with a nilpotent component G_0. If $-\alpha$ is the eigenvalue function of the irreducible G_0-module G_{-1} then $i\alpha$ is the unique eigenvalue function of the G_0-module G_i for each $i \in \mathbb{Z}$. The action of G_0 in G_{-1} is faithful. Note that the kernel of an irreducible representation coincides with the maximal ideal which acts in the module nilpotently. Hence G_0 acts faithfully in the compositional factors of the G_0-module G_i if and only if $i \not\equiv 0 \pmod p$. As we shall see now, the compositional factors of the G_0-modules G_i and G_j have the same dimension provided $i, j \not\equiv 0 \pmod p$.

LEMMA 1.4. *Let V_1, V_2 be two irreducible modules for a nilpotent Lie algebra H. Suppose that the corresponding eigenvalue functions α_1, α_2 are scalar multiples of each other. Then* $\dim V_1 = \dim V_2$.

Proof. Let ρ denote the representation of H in $V_1 \oplus V_2$ and $\tilde{H}$ the restricted subalgebra of the endomorphism Lie algebra $\mathrm{gl}(V_1 \oplus V_2)$ generated by $\rho(H)$. Then $\tilde{H}$ is a nilpotent restricted Lie algebra, V_1 and V_2 irreducible restricted $\tilde{H}$-modules. Denote by $\tilde{\alpha}_i$ the eigenvalue function of $\tilde{H}$ on V_i for $i = 1, 2$. Then $\alpha_i = \tilde{\alpha}_i \circ \rho$, $i = 1, 2$. It follows that the restrictions of $\tilde{\alpha}_1, \tilde{\alpha}_2$ to $\rho(H)$ are scalar multiples of each other. The $\tilde{H}$-module V_i is induced from a one-dimensional module of a restricted subalgebra P. Moreover, as is shown in [21], this assertion is valid for an arbitrary polarization P with respect to $\tilde{\alpha}_i$, this being a subalgebra of $\tilde{H}$ which satisfies $\tilde{\alpha}_i([P, P]) = 0$ and has maximal dimension among all subalgebras satisfying the previous property. The work [21] deals with irreducible representations of restricted solvable Lie algebras under assumption $p > 2$. However, the only fact actually needed in the proofs is that every non-abelian solvable Lie algebra has an abelian ideal not contained in the center. For nilpotent algebras this is valid for $p = 2$ as well. Since $[P, P] \subset [\tilde{H}, \tilde{H}] \subset \rho(H)$, the equality $\tilde{\alpha}_1([P, P]) = 0$ is equivalent to $\tilde{\alpha}_2([P, P]) = 0$. In other words, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ have the same polarizations. If P is any, then $\dim V_1 = p^{\dim \tilde{H}/P} = \dim V_2$.

LEMMA 1.5. *Let G be a graded Lie algebra satisfying* (g1)–(g4). *Suppose that G_0 is nilpotent. Then the simple graded Lie algebra S appearing in Theorem* 1.2 *satisfies* (g1)–(g4). *Furthermore, S_0 is nilpotent and* $[S_{-1}, S_1] = S_0$.

Proof. Since S_0 is a subalgebra of G_0, it is nilpotent. Since S is simple, we have $[S, S] = S$, whence $S_0 = [S_0, S_0] + [S_{-1}, S_1]$. It follows that $[S_{-1}, S_1]$ generates S_0. As it is an ideal, it coincides with S_0. It remains only to verify (g3). Let r be the smallest negative integer such that $S_r \neq 0$. Then S_r is a nontrivial irreducible S_0-module [26, Proposition 3.5]. Let $-\alpha, r\alpha$ be the eigenvalue functions of the G_0-modules G_{-1}, G_r. Their restrictions to S_0 give eigenvalue functions of the S_0 modules S_{-1}, S_r. Since S_r is nontrivial, $r \not\equiv 0 \pmod p$. By Lemma 1.4 the irreducible G_0-modules with eigenvalue functions $-\alpha$ and $r\alpha$ have the same dimension. As $G_r \cong B \otimes S_r$ is nonzero and $G_{-1} \cong B \otimes S_{-1}$ irreducible G_0-modules, we get therefore $\dim G_r \geq \dim G_{-1}$, whence $\dim S_r \geq \dim S_{-1}$. Applying Lemma 1.4 now to the irreducible representation of S_0, we conclude that S_{-1} is S_0-irreducible.

LEMMA 1.6. *Let H be a Lie algebra, $\overline{H}$ its factor algebra, U and V two H-modules, and $\varphi : U \times V \to \overline{H}$ an H-invariant bilinear mapping. Suppose that U is irreducible and $\varphi(U, V) = \overline{H}$. Denote by* $\operatorname{rad} V$ *the intersection of all maximal submodules of V. Then $\varphi(U, \operatorname{rad} V) \subset [\overline{H}, \overline{H}]$ and* $\dim \overline{H}/[\overline{H}, \overline{H}] \leq l$ *where l is the length of the factor module $V/\operatorname{rad} V$. In particular, if V contains a unique maximal submodule then the inequality* $\dim \overline{H}/[\overline{H}, \overline{H}] \leq 1$ *holds.*

Proof. Replacing $\overline{H}$ with $\overline{H}/[\overline{H}, \overline{H}]$ and φ with the composite of φ and the canonical projection $\overline{H} \to \overline{H}/[\overline{H}, \overline{H}]$, we may assume $\overline{H}$ to be abelian. This means that H acts trivially in $\overline{H}$. Then the H-module $\operatorname{Hom}(U, \overline{H}) \cong U^* \otimes \overline{H}$ is semisimple with all irreducible submodules isomorphic to U^*. The mapping φ induces an H-module homomorphism $\xi : V \to \operatorname{Hom}(U, \overline{H})$. Since $V/\ker \xi$ is semisimple, we have $\ker \xi \supset \operatorname{rad} V$. In other words, $\varphi(U, \operatorname{rad} V) = 0$. Any submodule of $\operatorname{Hom}(U, \overline{H})$ is a sum of certain irreducible submodules, whence it coincides with $\operatorname{Hom}(U, Q)$, where Q is a subspace of $\overline{H}$. If $\xi(V) = \operatorname{Hom}(U, Q)$, then $\varphi(U, V) = Q$. By hypotheses this implies $Q = \overline{H}$. Thus ξ is surjective. Therefore ξ induces an epimorphism of H-modules $V/\operatorname{rad} V \to \operatorname{Hom}(U, \overline{H})$. Hence the length of $V/\operatorname{rad} V$ is not less than that of $\operatorname{Hom}(U, \overline{H})$, i.e., $l \geq \dim \overline{H}$.

PROPOSITION 1.7. *Let G be a graded Lie algebra satisfying* (g1)–(g3). *Assume $[G_{-1}, G_1] = G_0$. Then one of the following holds*:

(1) *There exists a* nonzero *G_0-submodule $G'_1 \subset G_1$ with the property* $[[G_{-1}, G'_1], G'_1] = 0$;

(2) *There exists a* proper *G_0-submodule $G'_1 \subset G_1$ and a graded Lie algebra $\overline{G}'$ satisfying* (g1)–(g4) *such that $\overline{G}'_{-1} \cong G_{-1}$, $\overline{G}'_0 \cong G_0$, $\overline{G}'_1 \cong G'_1$, and $\overline{G}'_{-1}$ generates a simple ideal of $\overline{G}'$*;

(3) *G_1 contains a unique maximal G_0-submodule G_1'. In this case* $\dim G_0/[G_0, G_0] \le 1$ *and* $[G_{-1}, G_1'] \subset [G_0, G_0]$;

(4) *$G_1 = G_1' \oplus G_1''$ is a direct sum of two irreducible G_0-submodules. In this case* $\dim G_0/[G_0, G_0] \le 2$.

Proof. Suppose that G' is a homogeneous subalgebra of G such that $G'_{-1} = G_{-1}, G'_0 = G_0, [[G_{-1}, G_1'], G_1'] \ne 0$. Then G' satisfies (g1)–(g3) and $\overline{G}' := G'/M(G')$ satisfies (g1)–(g4). Since $M(G') \subset \bigoplus_{i<-1} G_i$, we have $\overline{G}_i' \cong G_i'$ for all $i \ge -1$. Furthermore, $A(\overline{G}')_+ \supset [[\overline{G}'_{-1}, \overline{G}_1'], \overline{G}_1'] \ne 0$. Denote by $\mathscr{C}(G')$ the centroid of $A(\overline{G}')$. By Theorem 1.2, $\mathscr{C}(G')$ is G_0-simple, all homogeneous components of $A(\overline{G}')$ are stable under $\mathscr{C}(G')$, and $A(\overline{G}') \cong \mathscr{C}(G') \otimes S'$ where $S' \subset A(\overline{G}')$ is a simple homogeneous subalgebra. Note that $A(\overline{G}')_{-1} = \overline{G}'_{-1} \cong G_{-1}$. In particular $S'_{-1} \ne 0$, whence $\mathscr{C}(G')$ acts faithfully on $A(\overline{G}')_{-1}$, and we may identify $\mathscr{C}(G')$ with a subalgebra of the endomorphism algebra $\operatorname{End} G_{-1}$. By the definition of the centroid, $[fx, y] = f[x, y]$ for all $f \in \mathscr{C}(G')$, $x, y \in A(\overline{G}')$, in particular for $x \in [G_{-1}, G_1'] \cong A(\overline{G}')_0$, and $y \in G_{-1} \cong A(\overline{G}')_{-1}$. Denoting by ρ the representation of G_0 in G_{-1}, we get $\rho(fx) = f \circ \rho(x)$ for $f \in \mathscr{C}(G')$, $x \in [G_{-1}, G_1']$. Note that $A(\overline{G}')$ is simple if and only if $\mathscr{C}(G') = k$.

Given a second homogeneous algebra $G'' \subset G$ such that $G''_{-1} = G_{-1}$, $G''_0 = G_0$, $[[G_{-1}, G_1''], G_1''] \ne 0$, we write $\mathscr{C}(G'') \subset \mathscr{C}(G')$ if there is an inclusion of corresponding subalgebras in $\operatorname{End} G_{-1}$. We claim that $\mathscr{C}(G'') \subset \mathscr{C}(G')$ whenever $G' \subset G''$. In proving this we may assume $G'' = G$. Replacing G and G' with the factor algebra $G/M(G)$ and its subalgebra $G'/G' \cap M(G)$, we may assume also that $M(G) = 0$, i.e., $\overline{G} = G$. We have $A(G') \subset A(G)$. By (g2), $G_- \subset A(G')$, whence $M(G') \subset A(G')$, and $A(G')/M(G') \cong A(\overline{G}')$. Denote $I := \{x \in A(G') | \mathscr{C}(G)x \subset A(G')\}$. The action of $\mathscr{C}(G)$ commutes with the adjoint representation of $A(G')$ in $A(G)$, whence I is an ideal of $A(G')$. Obviously, $G_{-1} \subset I$. By Lemma 1.3, $I = A(G')$, i.e., $A(G')$ is stable under $\mathscr{C}(G)$. Now $J := \mathscr{C}(G) \cdot M(G')$ is also an ideal of $A(G')$ which is contained in $\bigoplus_{i<-1} G_i$. By lemma 1.3, $J \subset M(G')$, i.e., $M(G')$ is stable under $\mathscr{C}(G)$. Thus $\mathscr{C}(G)$ operates in $A(\overline{G}')$, which gives rise to a homomorphism $\mathscr{C}(G) \to \mathscr{C}(G')$. Restricting to the components of degree -1, we see that the homomorphism is compatible with the natural embeddings of $\mathscr{C}(G), \mathscr{C}(G')$ into $\operatorname{End} G_{-1}$. Thus $\mathscr{C}(G) \subset \mathscr{C}(G')$.

Suppose that G_1', G_1'' are G_0-submodules of G_1 such that $G_1 = G_1' + G_1''$. Let G', G'' be the subalgebras of G generated by $G_{-1} \oplus G_0 \oplus G_1'$ and $G_{-1} \oplus G_0 \oplus G_1''$, respectively. We claim that one of the three possibilities occurs:

$$\text{(i)} \quad \mathscr{C}(G') = k, \qquad \text{(ii)} \quad \mathscr{C}(G'') = k,$$

$$\text{(iii)} \quad [[G_{-1}, G_1' \cap G_1''], G_1' \cap G_1''] = 0.$$

Indeed, assume (iii) does not hold. Then $\mathscr{C}(G' \cap G'')$ is a well defined commutative subalgebra of $\operatorname{End} G_{-1}$ which contains both $\mathscr{C}(G')$ and $\mathscr{C}(G'')$. In particular, the endomorphisms in $\mathscr{C}(G')$ and $\mathscr{C}(G'')$ commute with each other. Hence, given $f \in \mathscr{C}(G')$, $g \in \mathscr{C}(G'')$, $x \in [G_{-1}, G_1']$, we get

$$f[\rho(x), g] = [f\rho(x), g] = [\rho(fx), g] \in \mathscr{C}(G'')$$

since $\mathscr{C}(G'')$ is G_0-invariant. Since $\mathscr{C}(G')$ is also G_0-invariant, we deduce that $I := \{h \in \mathscr{C}(G'') | \mathscr{C}(G')h \subset \mathscr{C}(G'')\}$ is a G_0-invariant ideal of $\mathscr{C}(G'')$. We have proved that $[G_{-1}, G_1'] \cdot \mathscr{C}(G'') \subset I$. As $\mathscr{C}(G'')$ is G_0-simple, $I = \mathscr{C}(G'')$ or $I = 0$. In the first case $\mathscr{C}(G') \subset \mathscr{C}(G'')$. Now $[G_{-1}, G_1'] \cong A(\overline{G}')_0$ annihilates $\mathscr{C}(G')$; similarly, $[G_{-1}, G_1'']$ annihilates $\mathscr{C}(G'')$. We conclude that $\mathscr{C}(G')$ is annihilated by $G_0 = [G_{-1}, G_1] = [G_{-1}, G_1'] + [G_{-1}, G_1'']$. Since $\mathscr{C}(G')$ is G_0-simple, it follows $\mathscr{C}(G') = k$. Otherwise $I = 0$, in which case $[G_{-1}, G_1']$ annihilates $\mathscr{C}(G'')$, and we deduce similarly that $\mathscr{C}(G'') = k$.

We are ready now to complete the proof of the proposition. If G_1 contains a unique maximal G_0-submodule G_1' then, applying Lemma 1.6 to the multiplication mapping $G_{-1} \times G_1 \to G_0$, we get $\dim G_0/[G_0, G_0] \le 1$ and $[G_{-1}, G_1'] \subset [G_0, G_0]$. Otherwise $G_1 = G_1' + G_1''$ is a sum of two maximal submodules. If case (i) or (ii) of the preceding paragraph occurs then we are in case (2) of the proposition. Consider the remaining possibility (iii). If $G_1' \cap G_1'' \neq 0$ then we have case (1). Finally, if $G_1' \cap G_1'' = 0$ then $G_1 = G_1' \oplus G_1''$ and G_1', G_1'' are irreducible. By Lemma 1.6, then $\dim G_0/[G_0, G_0] \le 2$.

Two families of graded Lie algebras whose structure was determined by A. I. Kostrikin and I. R. Shafarevich [9, III, Theorem 2] are essential for the whole work on toral rank one. As cases of low characteristics are under consideration now, we need to make a few remarks. Suppose that G is a graded Lie algebra satisfying (g1), (g2) with $G_1 \neq 0$. The full Cartan prolongation $\tilde{G}$ of $G_- + G_0$ is an infinite dimensional graded Lie algebra containing G as a homogenous subalgebra and having the same homogenous components of degree ≤ 0. Denote by O_n the free divided power algebra in n indeterminates $x_1, \ldots, x_n$, by $\partial_1, \ldots, \partial_n$ its special derivations such that $\partial_i x_j = \delta_{ij}$ for $1 \le i, j \le n$. With each n-tuple of positive integers $\mathbf{m} = (m_1, \ldots, m_n)$ one associates a finite dimensional $\{\partial_1, \ldots, \partial_n\}$-invariant subalgebra $O_n(\mathbf{m}) \subset O_n$ generated by the elements $x_i^{(r)}$ with $i = 1, \ldots, n$, $r < p^{m_i}$. Cartan type Lie algebras related to $\mathbf{m}$ are certain Lie algebras of derivations of $O_n(\mathbf{m})$.

One of the families we are interested in is characterized by the conditions $\dim G_{-1} = \dim G_0 = 1$. Here $G_{-2} = [G_{-1}, G_{-1}] = 0$ and $\tilde{G} \cong W_1$. If $p > 2$, we have $G \cong \mathrm{sl}(2)$ or $W_1(m)$ for some m. If $p = 2$ then, using the formula $[f\partial, g\partial] = \partial(fg)\partial$ where $f, g \in O_1$, one verifies that ∂ and $x^{(p^m)}\partial$ generate the subalgebra $K_1'(m+1)$ of W_1. The arguments of [9] now ensure that $G \cong K_1'(m)$ or $W_1(m)$ for some m. The assumption $G_1 \neq 0$ implies $m > 1$.

The other family occurs with $G_{-2} = 0$, $\dim G_{-1} = 2$, $G_0 \cong \mathrm{sl}(G_{-1})$. Here $\tilde{G} \cong H_2$ consists of the derivations $\mathscr{D}(f) := \partial_1(f)\partial_2 - \partial_2(f)\partial_1$, $f \in O_2$. The multiplication is given by the rule $[\mathscr{D}(f), \mathscr{D}(g)] = \mathscr{D}([f, g])$ where $[f, g] := \partial_1(f)\partial_2(g) - \partial_2(f)\partial_1(g)$. Monomials $x_1^{(3)}, x_1^{(2)}x_2, x_1x_2^{(2)}, x_2^{(3)}$ represent a basis of $\tilde{G}_1$. If $p > 3$ then $\tilde{G}_1$ is G_0-irreducible. If $p = 3$ then $\tilde{G}_1$ contains a unique irreducible submodule $V := \mathscr{D}(kx_1^{(2)}x_2 + kx_1x_2^{(2)})$ and G_0 annihilates $\tilde{G}_1/V$. If $p = 2$ then the irreducible submodules of $\tilde{G}_1$ are of the form $\mathscr{D}(kfx_1 + kfx_2)$ where f is a linear combination of $x_1^{(2)}, x_2^{(2)}$. In this case $\tilde{G}_1$ is a sum of two irreducible submodules corresponding to $f = x_1^{(2)}$ and $f = x_2^{(2)}$. Lemma 4.2 further on describes a system of generators in a Lie algebra $H_2''(\mathbf{m})$. One sees that the arguments of [9] still work in low characteristics. Thus G is hamiltonian and $H_2''(\mathbf{m}) \subset G \subset H_2(\mathbf{m})$ for some $\mathbf{m} = (m_1, m_2)$. Suppose $p = 2$. If G_1 is an irreducible G_0-module, one which corresponds to some f as above, then $[G_{-1}, G_1]$ is spanned by $\mathscr{D}(x_1x_2), \mathscr{D}(f)$, and so is distinct from G_0. Thus $G_1 = \tilde{G}_1$, i.e., $m_1, m_2 > 1$, whenever $[G_{-1}, G_1] = G_0$.

We note one consequence of the preceding fact. If G satisfies case (3) of Theorem 1.1 then $\dim G_1 > \dim G_{-1}$. Indeed, replacing G with $G'/M(G')$ where G' is the subalgebra of G generated by $G_{-1} + G_0 + G_1$, we may assume that G satisfies (g1)–(g4) and that G_+ is generated by G_1. Condition $[G_{-2}, G_1] = 0$ then yields $[G_{-2}, G_+] = 0$, whence $G_{-2} = 0$, and so G is hamiltonian. Condition $G_0 = [G_{-1}, G_1]$ implies $\dim G_1 = 4$.

Proof of Theorem 1.1. We proceed by induction on $\dim G_1$. If $G' \subset G$ is a homogeneous subalgebra such that $G'_{-1} = G_{-1}$, $G'_0 = G_0$, $G'_1 \neq 0$ then $\overline{G}' := G'/M(G')$ satisfies (g1)–(g4) and $A(\overline{G}')_+ \neq 0$ because G_0 is nilpotent. This shows, in particular, that case (1) of Proposition 1.7 cannot occur. Actually $[[G_{-1}, G'_1], G'_1] = G'_1$ because $[G_{-1}, G'_1] \neq 0$ and G_0 acts faithfully in the compositional factors of the G_0-module G_1. In other words, $A(\overline{G}')_1 = \overline{G}'_1$. By Theorem 1.2, $A(\overline{G}') \cong B' \otimes S'$, $A(\overline{G}')_i \cong B' \otimes S'_i$ for all i, where B' is a commutative associative algebra, S' a simple graded Lie algebra which satisfies, according to Lemma 1.5, the hypotheses of Theorem 1.1. Thus, if $G'_1 \neq G_1$, either $\dim S'_{-1} = 1$ or $p = 2$, $\dim S'_{-1} = 2$, $S'_0 \cong \mathrm{sl}(S'_{-1})$. We have also $[S'_{-2}, S'_1] = 0$, whence $[\overline{G}'_{-2}, \overline{G}'_1] = [A(\overline{G}')_{-2}, A(\overline{G}')_1] = 0$, and $[G_{-2}, G'_1] = 0$. Note that in the case when G'_1 is G_0-irreducible we have $\dim G'_1 = \dim G_{-1}$ by Lemma 1.4, hence $\dim S'_1 = \dim S'_{-1}$. According to the remarks preceding the proof this holds only when $\dim S'_{-1} = 1$. In this case $\dim S'_0 = 1$ as well, hence $[G_{-1}, G'_1] \cong A(\overline{G}')_0 \cong B' \otimes S'_0$ is an abelian ideal of G_0 and $\rho([G_{-1}, G'_1]) \cong B'$ is a subalgebra of the associative algebra $\operatorname{End} G_{-1}$, where ρ denotes the representation of G_0 in $G_{-1} \cong B' \otimes S'_{-1}$.

We shall examine now different possibilities occurring in Proposition 1.7. Suppose (2) holds. Then $G_{-1} \cong \overline{G}'_{-1} = S'_{-1}$. Thus $\dim G_{-1} = 1$ or $p = 2$, $\dim G_{-1} = 2$, $G_0 \supset \mathrm{sl}(G_{-1})$. Since G_0 is nilpotent, $G_0 = \mathrm{sl}(G_{-1})$ in the latter case. Suppose (3) holds. Since G_0 is nilpotent, the property $\dim G_0/[G_0, G_0] \leq 1$ implies $\dim G_0 = 1$. The G_0-irreducibility of G_{-1} forces $\dim G_{-1} = 1$. Suppose (4) holds. Then $G_0 = [G_{-1}, G'_1] + [G_{-1}, G''_1]$ is a sum of two abelian ideals. Hence $[G_0, G_0]$ is contained in the center of G_0. Since G_0 acts faithfully and irreducibly in G_{-1}, its center consists of scalar transformations of G_{-1} and, in particular, is one-dimensional. It follows that G_0 is a Heisenberg algebra. The property $\dim G_0/[G_0, G_0] \leq 2$ implies $\dim G_0 = 3$. Choose $X \in \rho([G_{-1}, G'_1])$ and $Y \in \rho([G_{-1}, G''_1])$ such that $[X, Y] = \mathrm{Id}$. Then $[X^2, Y] = 2X$ is not a scalar transformation, hence $X^2 \notin \rho([G_{-1}, G'_1])$, unless $p = 2$. Since $\rho([G_{-1}, G'_1])$ must be a subalgebra of $\operatorname{End} G_{-1}$, we get $p = 2$. Since G_{-1} is a nontrivial irreducible G_0-module, we get $\dim G_{-1} = 2$ and $G_0 \cong \mathrm{sl}(G_{-1})$.

Note also that the case $p = 2, \dim G_{-1} = 2$ can occur only when G_1 contains at least two different maximal submodules, say G'_1 and G''_1. As we have noted, $[G_{-2}, G'_1] = [G_{-2}, G''_1] = 0$. Since $G_1 = G'_1 + G''_1$, we conclude $[G_{-2}, G_1] = 0$.

2. RESULTS ON SPENCER HOMOLOGY

Spencer homology is a common tool to obtain information about a filtered deformation L of a graded Lie algebra G (see [5, 6, 16]). We want to find certain conditions on G which ensure that $[L, L] \neq L$, and so G cannot occur as a graded algebra associated with a filtration of a simple Lie algebra. We assume here that the gradation of G has depth 1. Then G_{-1} is a commutative subalgebra of G and we regard G as a G_{-1}-module with respect to adjoint representation. The spaces of n-cocycles $Z^n(G_{-1}, G)$, n-coboundaries $B^n(G_{-1}, G)$, and cohomology classes $H^n(G_{-1}, G)$ inherit $\mathbb{Z}$-gradations from G. Denote by $Z^{ni}(G)$ the subspace of $Z^n(G_{-1}, G)$ consisting of cocycles with values in G_i, by $B^{ni}(G)$ its intersection with $B^n(G_{-1}, G)$, and by $H^{ni}(G)$ the factor $Z^{ni}(G)/B^{ni}(G)$. If $G' \subset G$ is a homogeneous subalgebra with $G'_{-1} = G_{-1}$, then inclusions $Z^{ni}(G') \subset Z^{ni}(G)$, $B^{ni}(G') \subset B^{ni}(G)$ induce a linear mapping $H^{ni}(G') \to H^{ni}(G)$. It is a homomorphism of G_0-modules, and so its cokernel is a G_0-module, provided that G' is G_0-invariant.

THEOREM 2.1. *Let $L = L_{-1} \supset L_0 \supset \cdots$ be a filtered Lie algebra, G its associated graded algebra, $G' = G_{-1} \oplus G'_0$ where G'_0 is an ideal of G_0*

containing $[G_0, G_0] + [G_{-1}, G_1]$. *Suppose that* $H^0(G_0, H^{2,-1}(G)) = 0$. *Then there exists a subspace* $P \subset L$ *such that* $L = P + L_0$, $L_1 = P \cap L_0$ *and* $[P, P] \subset L_0$. *Furthermore*:

(1) *If* $H^1(G_0, H^{10}(G)) = 0$ *then there exists* P *as above such that* $N_{L_0}(P) + L_1 = L_0$ *where* $N_{L_0}(P)$ *denotes the normalizer of* P *in* L_0. *If in addition* $H^{01}(G) = 0$ (*which always holds provided* G *satisfies* (g1)) *then* $N_{L_0}(P) \cap L_1 = L_2$;

(2) *Denote by* K^1 *the cokernel of the mapping* $H^{10}(G') \to H^{10}(G)$. *If* $H^1(G_0, K^1) = 0$ *then* $\mathrm{gr}_0[L, L_0] \subset G'_0$;

(3) *Denote by* K^2 *the cokernel of the mapping* $H^{20}(G') \to H^{20}(G)$. *If* $H^1(G_0, K^1) = 0$ *and* $H^0(G_0, K^2) = 0$ *then* $\mathrm{gr}_0[L, L] \subset G'_0$.

Proof. Let $\pi_i : L_i \to G_i := L_i/L_{i+1}$, $i \geq -1$, be the canonical projections and $\sigma : G \to L$ a linear mapping such that $\pi_i \sigma|_{G_i} = \mathrm{id}$ and, in particular, $L_i = \sigma(G_i) \oplus L_{i+1}$ for each i. The choice of σ is not unique. Using information on Spencer homology groups, we shall adjust σ in order to obtain some special properties of the subspaces $\sigma(G_{-1})$, $\sigma(G_0)$ with respect to the multiplication in L. There exist skewsymmetric bilinear mappings $\varphi^r : G \times G \to G$, $r \geq 1$, such that $\varphi^r(G_i, G_j) \subset G_{i+j+r}$ and

$$[\sigma(x), \sigma(y)] \equiv \sigma\big([x, y] + \varphi^1(x, y) + \cdots + \varphi^r(x, y)\big) \qquad \big(\mathrm{mod}\ L_{i+j+r+1}\big)$$

for all $i, j \geq -1$, $x \in G_i$, $y \in G_j$. These mappings satisfy the well-known deformation equations. In particular,

$$d\varphi^1 = 0, \qquad d\varphi^2 = \varphi^1 \overline{\wedge}\, \varphi^1,$$

where

$$\big(\varphi^1 \overline{\wedge}\, \varphi^1\big)(x, y, z) = \varphi^1\big(\varphi^1(x, y), z\big) + \varphi^1\big(\varphi^1(y, z), x\big) + \varphi^1\big(\varphi^1(z, x), y\big)$$

for $x, y, z \in G$ and d is the differential of the standard cochain complex for the Lie algebra G and its adjoint module. If $\tau : G \to L$ is a linear mapping such that $\tau(G_i) \subset L_{i+r}$ for all i then the replacement of σ with $\sigma + \tau$ does not affect $\varphi^1, \ldots, \varphi^{r-1}$, while φ^r changes by a coboundary. Denote by φ^r_{ij} the restriction of φ^r to $G_i \times G_j$.

First of all, $\varphi^1_{-1,-1} \in Z^{2,-1}(G)$. Its cohomology class $\overline{\varphi}^1_{-1,-1} \in H^{2,-1}(G)$ is an invariant of L called first order structure constant [5]. The equation $d\varphi^1(x, y, z) = 0$ for $x \in G_0$, $y, z \in G_{-1}$ gives

$$\big(x \cdot \varphi^1_{-1,-1}\big)(y, z) = \big[y, \varphi^1_{0,-1}(x, z)\big] - \big[z, \varphi^1_{0,-1}(x, y)\big]. \qquad (*)$$

It follows that $x \cdot \varphi^1_{-1,-1}$ is a coboundary for each $x \in G_0$. Hence G_0 annihilates $\overline{\varphi}^1_{-1,-1}$. By hypotheses, this implies that $\varphi^1_{-1,-1}$ is a coboundary. Adjusting σ on G_{-1} we can achieve $\varphi^1_{-1,-1} = 0$. This means that $[\sigma(G_{-1}), \sigma(G_{-1})] \subseteq L_0$. We may take $P = \sigma(G_{-1}) + L_1$.

Suppose that $H^1(G_0, K^1) = 0$. Put $L'_0 = \pi_0^{-1}(G'_0)$. This is an ideal of L_0. We claim that there exists a subspace $Q \subset L$ such that $Q + L_0 = L$, $Q \cap L_0 = L'_0$, and $N_{L_0}(Q) + L_1 = L_0$. Actually we shall prove this under assumptions weaker than those in the statement of the theorem. Namely, we do not require the inclusion $G'_0 \supset [G_{-1}, G_1] + [G_0, G_0]$. Define $\theta : G_0 \to \mathrm{Hom}(G_{-1}, G_0)$ by the formula $\theta(x)(y) = \varphi^1_{0,-1}(x, y)$ for $x \in G_0$, $y \in G_{-1}$. Since $\varphi^1_{-1,-1} = 0$, it follows from $(*)$ that $\theta(G_0) \subset Z^{10}(G)$. Let $\bar\theta : G_0 \to H^{10}(G)$ denote the composite of θ and the projection $Z^{10}(G) \to H^{10}(G)$, and $\bar{\bar\theta} : G_0 \to K^1$ the composite of $\bar\theta$ and the projection $H^{10}(G) \to K^1$. Taking $x, y \in G_0$, $z \in G_{-1}$ in the equation $d\varphi^1(x, y, z) = 0$, we get

$$(x \cdot \theta(y) - y \cdot \theta(x) - \theta([x, y]))(z) = \left[\varphi^1_{00}(x, y), z\right],$$

whence $x \cdot \theta(y) - y \cdot \theta(x) - \theta([x, y]) \in B^{10}(G)$. It follows that $\bar\theta$, hence also $\bar{\bar\theta}$, are 1-cocycles. By assumption $\bar{\bar\theta} = d'\bar{\bar\eta}$ for some $\bar{\bar\eta} \in K^1$. We denote by d' and d'' the differentials of the standard cochain complexes for the Lie algebras G_0 and G_{-1}, respectively. Pick out a preimage $\eta \in Z^{10}(G)$ of $\bar{\bar\eta}$. Then $(\theta - d'\eta)(G_0)$ is contained in the kernel of the projection $Z^{10}(G) \to K^1$, which is $B^{10}(G) + Z^{10}(G')$. Hence there exists a linear mapping $\zeta : G_0 \to G_1$ such that $(\theta - d'\eta - d'' \circ \zeta)(G_0) \subset Z^{10}(G')$. In other words, $\varphi^1_{0,-1}(x, y) - (x \cdot \eta)(y) - [y, \zeta(x)] \in G'_0$ for all $x \in G_0$, $y \in G_{-1}$. Let $\psi : G \to G$ be a homogeneous linear endomorphism of degree 1 such that $\psi|_{G_{-1}} = \eta$ and $\psi|_{G_0} = \zeta$. Then $d\psi(x, y) = [x, \eta(y)] - [y, \eta(x)] = 0$ for $x, y \in G_{-1}$ and $d\psi(x, y) = (x \cdot \eta)(y) - [y, \zeta(x)]$ for $x \in G_0$, $y \in G_{-1}$. Adjusting σ, we can replace φ^1 with $\varphi^1 - d\psi$. Then we achieve $\varphi^1(G_0, G_{-1}) \subset G'_0$, retaining $\varphi^1(G_{-1}, G_{-1}) = 0$. Put $Q = \sigma(G_{-1}) + L'_0$ and note that $\sigma(G_0)$ normalizes Q.

If $H^1(G_0, H^{10}(G)) = 0$ then we can apply the preceding with $G'_0 = 0$, $L'_0 = L_1$. Then $P := Q$ is the desired subspace. Since $G_{-1} = \mathrm{gr}_{-1} P$, we have $[\mathrm{gr}_1 N_{L_0}(P), G_{-1}] \subset \mathrm{gr}_0\, P = 0$. If $H^{01}(G) = 0$ then $\mathrm{gr}_1 N_{L_0}(P) = 0$.

Suppose further that $G'_0 \supset [G_{-1}, G_1] + [G_0, G_0]$. Then $L'_0 \supset [L_{-1}, L_1] + [L_0, L_0]$. In particular $N_{L_0}(Q) \supset L_1$, whence $N_{L_0}(Q) = L_0$. It follows $[L_0, L] = [L_0, L_0 + Q] \subset Q$. Thus $\mathrm{gr}_0[L_0, L] \subset \mathrm{gr}_0 Q = G'_0$.

Suppose that $H^0(G_0, K^2) = 0$. If $x, y, z \in G_{-1}$ then $(\varphi^1 \overline{\wedge} \varphi^1)(x, y, z) = 0$, whence $(d\varphi^2)(x, y, z) = 0$. This shows that $\varphi^2_{-1,-1} \in Z^{20}(G)$. Taking $x \in G_0$, $y, z \in G_{-1}$, we get

$$\begin{aligned}(x \cdot \varphi^2_{-1,-1})(y, z) = \left[y, \varphi^2_{0,-1}(x, z)\right] - \left[z, \varphi^2_{0,-1}(x, y)\right]& \\ + \left(\varphi^1 \overline{\wedge} \varphi^1\right)(x, y, z).&\end{aligned}$$

Note that $(\varphi^1 \overline{\wedge} \varphi^1)(x, y, z) \in G'_0$. We conclude $G_0 \cdot \varphi^2_{-1,-1} \subset B^{20}(G) + Z^{20}(G')$. The right hand side of this inclusions is the kernel of the projection $Z^{20}(G) \to H^{20}(G) \to K^2$. Hence G_0 annihilates the image of $\varphi^2_{-1,-1}$ in K^2. It follows that this image is zero, i.e., $\varphi^2_{-1,-1} \in B^{20}(G) + Z^{20}(G')$. Thus there exists a linear mapping $\xi : G_{-1} \to G_1$ such that $\varphi^2_{-1,-1} - d''\xi \in Z^{20}(G')$. Extend ξ to a homogeneous linear endomorphism of G of degree 2. Adjusting σ, we can replace φ^2 with $\varphi^2 - d\xi$, retaining φ^1. Then we achieve $\varphi^2_{-1,-1} \in Z^{20}(G')$, i.e., $\varphi^2(G_{-1}, G_{-1}) \subset G'_0$. It follows $[\sigma(G_{-1}), \sigma(G_{-1})] \subset L'_0$. Since $L = L_0 + \sigma(G_{-1})$, we have $[L, L] \subset [L, L_0] + [\sigma(G_{-1}), \sigma(G_{-1})] \subset Q$. Thus $\mathrm{gr}_0[L, L] \subset G'_0$.

LEMMA 2.2. *Let $L = L_{-1} \supset L_0 \supset \cdots$ be a filtered Lie algebra, G its associated graded algebra, G'_0 an ideal of G_0. Suppose that G_{-1} is an irreducible G_0-module, $\mathrm{gr}_0[L, L_0] \subset G'_0$ and $G'_0 + \mathrm{gr}_0[L, L] = G_0$. Then either $G_0 = G'_0$ or $\dim G_0/G'_0 = 1$ and there exists a G_0-invariant nondegenerate skewsymmetric bilinear form on G_{-1}.*

Proof. The Lie algebra L_0 acts trivially in $V := L/[L, L_0]$. The composite $\alpha : L \times L \to V$ of the multiplication $L \times L \to L$ and the canonical projection $L \to V$ is an L_0-invariant skewsymmetric bilinear mapping. As $\alpha(L, L_0) = 0$, it induces a G_0-invariant skewsymmetric bilinear mapping $\beta: G_{-1} \times G_{-1} \to V$, where G_0 acts trivially in V. For each linear form $\lambda \in V^*$ the composite $\lambda \circ \beta$ is a G_0-invariant skewsymmetric bilinear form on G_{-1}. Since G_{-1} is G_0-irreducible, the space of G_0-invariant bilinear forms on G_{-1} is at most one-dimensional. It follows that $\dim \beta(G_{-1}, G_{-1}) \leq 1$. This means that the codimension of $[L, L_0]$ in $[L, L]$, hence also that of $[L, L_0] \cap L_0$ in $[L, L] \cap L_0$, is not greater than 1. Applying the projection $L_0 \to G_0$, we see that this is valid also for the codimensions of $\mathrm{gr}_0[L, L_0]$ in $\mathrm{gr}_0[L, L]$ and G'_0 in $G'_0 + \mathrm{gr}_0[L, L] = G_0$. If $G'_0 \neq G_0$ then $[L, L_0] \neq [L, L]$, whence $\beta \neq 0$. Then there exists a nonzero G_0-invariant skewsymmetric bilinear form on G_{-1}.

In the following sections we shall meet the hypotheses of Theorem 2.1 in two different cases. One of these is fairly well known. Let us keep the notations of the theorem.

LEMMA 2.3. *Suppose that G_{-1} is an irreducible faithful G_0-module and G_0 has a nontrivial center. Then $H^0(G_0, H^{2,-1}(G))$, $H^1(G_0, H^{10}(G))$, and $H^1(G_0, K^1)$ all vanish. If $p > 2$ then $H^0(G_0, K^2) = 0$ as well.*

Proof. Let z be the element of the center of G_0 which acts as minus identity transformation on G_{-1}. Then z acts as identity transformation in $H^{2,-1}(G)$, $H^{10}(G)$, K^1. It acts as the multiplication by 2 in K^2. The result is immediate now.

The other case is when we can prove $Z^{10}(G) = Z^{10}(G')$ and $Z^{20}(G) = Z^{20}(G')$. Note that $K^1 = 0$ (respectively $K^2 = 0$) whenever the former (respectively the latter) equality holds.

LEMMA 2.4. *Let ∂ be a nilpotent derivation of a commutative associative algebra B. Suppose that B is ∂-simple. Then there exists an isomorphism of algebras $B \cong O_1(m)$ for some $m > 0$ such that the induced action of ∂ in $O_1(m)$ is given by the rule $x^{(r)} \mapsto x^{(r-1)}$, $0 < r < p^m$.*

Proof. We have $B = k + \mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of B. Put $\mathfrak{h} := \{D \in P(k\partial) | D\mathfrak{m} \subset \mathfrak{m}\}$ where $P(k\partial)$ denotes the universal p-envelope of the one-dimensional Lie algebra $k\partial$. According to [18, I, Theorem 4.1] there is a ∂-invariant isomorphism $B \cong F(k\partial, \mathfrak{h}) := \operatorname{Hom}_{u(\mathfrak{h})}(U(k\partial), k)$ where $U(k\partial)$ is the universal enveloping algebra of $k\partial$ and $u(\mathfrak{h})$ its subalgebra generated by $\mathfrak{h}$. There exists $m > 0$ such that $\partial^{p^m} \in \mathfrak{h}$. If m is the smallest number with this property then ∂^{p^i}, $0 \le i < m$, are linearly independent modulo $\mathfrak{h}$, i.e., ∂^{p^m} generates $\mathfrak{h}$ as a restricted Lie algebra. The elements ∂^r, $0 \le r < p^m$, form a basis of $U(k\partial)$ over $u(\mathfrak{h})$. Then a basis of $F(k\partial, \mathfrak{h})$ over k is $y^{(r)}$, $0 \le r < p^m$, where $y^{(r)}(\partial^s) = \delta_{rs}$, $0 \le r, s < p^m$. We have $\partial y^{(r)} = y^{(r-1)}$, and the assignment $x^{(r)} \mapsto y^{(r)}$ defines the desired isomorphism of $O_1(m)$ onto $F(k\partial, \mathfrak{h})$.

LEMMA 2.5. *If H is a nilpotent Lie algebra and V its irreducible module then H acts nilpotently on* $\operatorname{End} V$.

Proof. Each element $x \in H$ has a single eigenvalue on V. Hence 0 is a single eigenvalue of x on $\operatorname{End} V$.

PROPOSITION 2.6. *Let B be a G_0-invariant commutative subalgebra of the associative algebra* $\operatorname{End} G_{-1}$. *Assume that B is G_0-simple, G_0 acts nilpotently on B, and G_{-1} is a free B-module (the last condition is actually a consequence of G_0-simplicity of B). Denote $G_0' := \rho^{-1}(\operatorname{End}_B G_{-1})$ where ρ is the representation of G_0 on G_{-1}. Put $G' := G_{-1} \oplus G_0'$. Then*

(1) $Z^{10}(G) = Z^{10}(G')$ *unless* $p = 2$, $\operatorname{rk}_B G_{-1} = 1$;

(2) *Assume in addition that* $\dim G_0/G_0' \le 1$. *Then* $Z^{20}(G) = Z^{20}(G')$ *unless one of the following holds*:

(a) $p = 3$, $\dim B = 3$, $\operatorname{rk}_B G_{-1} \le 2$. *If* $\operatorname{rk}_B G_{-1} = 2$ *then* $\rho(G_0') \not\subset \mathrm{sl}_B(G_{-1})$;

(b) $p = 2$, $\dim B = 2$;

(c) $p = 2$, $\dim B = 4$, $\operatorname{rk}_B G_{-1} = 1$.

Here $\mathrm{sl}_B(G_{-1}) = \{\varphi \in \operatorname{End}_B G_{-1} | \operatorname{tr}_B \varphi = 0\}$ where tr_B denotes the trace function $\operatorname{End}_B G_{-1} \to B$.

Proof. Let $\pi : G_0 \to \operatorname{Der} B$ designate the natural homomorphism. So $\pi(x)(f) = [\rho(x), f]$ for $x \in G_0$, $f \in B$. In particular, $G_0' = \ker \pi$.

(1) Suppose first that $\operatorname{rk}_B G_{-1} > 1$. Let $\varphi \in Z^{10}(G)$, i.e., $\varphi : G_{-1} \to G_0$ is a linear mapping satisfying $[\varphi(u), v] = [\varphi(v), u]$ for all $u, v \in G_{-1}$. Put $\psi = \pi \circ \varphi$. Given $f, g \in B$, we have

$$\psi(u)(g)v = [\varphi(u), gv] - g[\varphi(u), v] = [\varphi(gv), u] - g[\varphi(v), u],$$

$$(\psi(fu) - f\psi(u))(g)v = (\psi(gv) - g\psi(v))(f)u.$$

If u is contained in a basis of G_{-1} as a B module, then we may take v to be a different element of this basis. It follows $\psi(fu) = f\psi(u)$. Thus ψ is B-linear. Actually this fact is explicitly mentioned in the proof of [11, Proposition 4.1], although under different settings. We have to prove that $\varphi(G_{-1}) \subset G_0'$, i.e., $\psi = 0$. By the above $\psi(G_{-1}) \subset T$ where $T := \{D \in \pi(G_0) | BD \subset \pi(G_0)\}$ is the largest B-submodule of $\operatorname{Der} B$ contained in $\pi(G_0)$. Obviously T is an ideal of $\pi(G_0)$. Suppose $T \neq 0$. Then the elements $D(f)$ with $D \in T, f \in B$ generate a nonzero G_0-invariant ideal I of B. The G_0-simplicity of B ensures $I = B$. Since B is local (this is a well-known consequence of differential simplicity), there exist $D \in T$ and $f \in B$ such that $h := D(f)$ is invertible in B. Then $D' := h^{-1}fD \in \pi(G_0)$ and $D'(f) = f$, i.e., D' is not nilpotent, a contradiction.

The case $\operatorname{rk}_B G_{-1} = 1$, $p > 2$ is an easy consequence of Theorem 1.1 (another path is to follow [11, Proposition 4.1]). Consider the Lie algebra $\tilde{G}_0 := B + \rho(G_0)$ of linear transformations of G_{-1}. If $V \subset G_{-1}$ is a $\tilde{G}_0$-invariant subspace, then $V = IG_{-1}$ where I is a G_0-invariant ideal of B. It follows $V = 0$ or $V = G_{-1}$, i.e., G_{-1} is an irreducible $\tilde{G}_0$-module. Next, B is an abelian ideal of $\tilde{G}_0$ on which $\tilde{G}_0$ acts nilpotently. Since $\operatorname{End}_B G_{-1} = B$, the kernel of representation $\tilde{G}_0 \to \operatorname{Der} B$ coincides with B. It follows that $\tilde{G}_0/B$ is nilpotent, whence so is $\tilde{G}_0$ as well. Let $\tilde{G} = \bigoplus_{i \geq -1} \tilde{G}_i$ be the full Cartan prolongation of the pair $(G_{-1}, \tilde{G}_0)$. We may identify $\tilde{G}_1$ with $Z^{10}(\tilde{G})$ so that $[\varphi, u] = \varphi(u)$ for $\varphi \in \tilde{G}_1$, $u \in G_{-1}$. Note that $\operatorname{Hom}_B(G_{-1}, B) \subset \tilde{G}_1$, whence $B \subset [G_{-1}, \tilde{G}_1]$. We cannot claim that $\dim \tilde{G} < \infty$. However, $G_{-1} \oplus \tilde{G}_0 \oplus \tilde{G}_1$ generates a finite dimensional subalgebra. Indeed, it is contained in the subalgebra of $\tilde{G}$ consisting of the elements x such that $(\operatorname{ad} u)^p x = 0$ for all $u \in G_{-1}$. The latter is obviously finite dimensional (this argument was communicated to me by Yu. B. Ermolaev). We replace $\tilde{G}$ with this subalgebra and apply Theorem 1.2 to it. Thus $A(\tilde{G}) \cong B' \otimes S$ where S satisfies the hypotheses of Theorem 1.1 according to Lemma 1.5. Hence $\dim S_{-1} = \dim S_0 = 1$. We see that G_{-1} is free of rank 1 over B' and $[G_{-1}, \tilde{G}_1] = A(\tilde{G})_0 = B'$. Since $B \subset B'$ and $\operatorname{rk}_B G_{-1} = 1$, it follows $B = B'$. Thus $[G_{-1}, \tilde{G}_1] = B$. If $\varphi : G_{-1} \to G_0$ is an element of $Z^{10}(G)$ then $\rho \circ \varphi : G_{-1} \to \tilde{G}_0$ is an element of $Z^{10}(\tilde{G})$. Hence $\rho(\varphi(G_{-1})) \subset B$, i.e., $\varphi(G_{-1}) \subset G_0'$.

(2) Since the case $G_0 = G_0'$ is trivial, we may assume that $\dim \pi(G_0) = \dim G_0/G_0' = 1$. Let $\pi(G_0) = k\partial$. By Lemma 2.4, $B \cong O_1(m)$ and ∂ acts on $O_1(m)$ canonically. Let $\varphi \in Z^{20}(G)$. Define $\psi : G_{-1} \times G_{-1} \to k$ setting $\pi(\varphi(u,v)) = \psi(u,v)\partial$ for $u, v \in G_{-1}$. We have to show that $\psi = 0$ except for the cases listed in the statement of Proposition 2.6. Choose a basis $e_1, \dots, e_n$ of G_{-1} over B. Define $D \in \operatorname{End} G_{-1}$ by the rule $D(fe_i) = \partial(f)e_i$ for $f \in B$, $1 \le i \le n$. We have $\rho(\varphi(u,v)) = \psi(u,v)D + \varphi'(u,v)$ where $\varphi'(u,v) \in \operatorname{End}_B G_{-1}$. The condition that φ is a cocycle means $[\varphi(u,v),w] + [\varphi(v,w),u] + [\varphi(w,u),v] = 0$ for all $u,v,w \in G_{-1}$. Take $u = fe_i$, $v = ge_j$, $w = he_k$ where $f,g,h \in B$, $1 \le i,j,k \le n$, and rewrite this as

$$\begin{aligned}\psi(fe_i, ge_j)\partial(h)e_k + \psi(ge_j, he_k)\partial(f)e_i + \psi(he_k, fe_i)\partial(g)e_j \\ + h\varphi'(fe_i, ge_j)(e_k) + f\varphi'(ge_j, he_k)(e_i) + g\varphi'(he_k, fe_i)(e_j) = 0.\end{aligned} \tag{$*$}$$

Now B contains an element x such that $\partial(x) = 1$. Substitute first 1, then x for h above and subtract the former multiplied by x from the latter. We get

$$\begin{aligned}\psi(fe_i, ge_j)e_k = {} & \partial(f)\big(\psi(xe_k, ge_j) - x\psi(e_k, ge_j)\big)e_i \\ & - \partial(g)\big(\psi(xe_k, fe_i) - x\psi(e_k, fe_i)\big)e_j \\ & + f\big(\varphi'(xe_k, ge_j) - x\varphi'(e_k, ge_j)\big)(e_i) \\ & - g\big(\varphi'(xe_k, fe_i) - x\varphi'(e_k, fe_i)\big)(e_j).\end{aligned}$$

Comparing coefficients of e_k, we find an expression for $\psi(fe_i, ge_j)$. We see that there exist linear mappings $\alpha_{ij} : B \to B$, $\beta_{ij} : B \to B$, $1 \le i,j \le n$, such that

$$\psi(fe_i, ge_j) = f\alpha_{ij}(g) - g\alpha_{ji}(f) + \partial(f)\beta_{ij}(g) - \partial(g)\beta_{ji}(f).$$

Furthermore, if there exists an index $k \ne i,j$ then we may take $\beta_{ij} = \beta_{ji} = 0$. This holds for each pair i,j when $n \ge 3$ and for $i = j$ when $n = 2$. If $n = 2$ and $i \ne j$ then we get an expression with $\beta_{ji} = 0$ taking $k = i$.

Now we shall use the condition that $\psi(fe_i, ge_j)$ is a scalar. More generally, let $\tilde{\psi} : B \times B \to B$ be a bilinear mapping such that $\tilde{\psi}(f,g) \in k$, hence $\partial(\tilde{\psi}(f,g)) = 0$, for all $f, g \in B$. Suppose that

$$\tilde{\psi}(f,g) = f\alpha(g) + g\alpha'(f) + \partial(f)\beta(g) + \partial(g)\beta'(f)$$

for some linear endomorphisms $\alpha, \alpha', \beta, \beta' : B \to B$. Identify B with a subalgebra of End B. Then End B is free over B with ∂^r, $0 \le r < p^m$, as its basis. Expanding $\alpha, \alpha', \beta, \beta'$ over this basis, we get

$$\tilde{\psi}(f, g) = \sum_{0 \le r, s < p^m} a_{rs} \partial^r(f) \partial^s(g),$$

where $a_{rs} \in B$, $a_{rs} = 0$ whenever $r, s > 1$. Now

$$\partial\big(\tilde{\psi}(f, g)\big) = \sum_{0 \le r, s < p^m} c_{rs} \partial^r(f) \partial^s(g),$$

where $c_{rs} = \partial(a_{rs}) + a_{r-1,s} + a_{r,s-1}$; we set $a_{-1,s} = 0$ and $a_{r,-1} = 0$. Since ∂^r, $0 \le r < p^m$, are linearly independent over B, the condition $\partial(\tilde{\psi}(f, g)) = 0$ implies $c_{rs} = 0$, i.e.,

$$\partial(a_{rs}) + a_{r-1,s} + a_{r,s-1} = 0, \qquad 0 \le r, s < p^m. \tag{$**$}$$

Take here $r = 2$ and obtain $a_{1s} = 0$ whenever $s > 2$. Next take $r = 1$ and obtain $a_{0s} = 0$ whenever $s > 3$. Suppose that $a_{0s} \ne 0$ for some s. Let t be the largest integer such that $a_{0t} \ne 0$. In particular, $t \le 3$. If $t < p^m - 1$ then we could take $r = 0$, $s = t + 1$, and obtain from $(**)$ $a_{0t} = 0$, a contradiction. Hence $t = p^m - 1$, and so $p^m \le 4$. Suppose that $a_{0s} = 0$ for all s, but $a_{1s} \ne 0$ for some s. Let t be the largest integer such that $a_{1t} \ne 0$. In particular, $t \le 2$. If $t < p^m - 1$ then we could take $r = 1$, $s = t + 1$ and obtain from $(**)$ $a_{1t} = 0$, a contradiction. Hence $t = p^m - 1$, and so $p^m \le 3$. We can investigate similarly a_{r0} and a_{r1}. Thus all coefficients vanish, i.e., $\tilde{\psi} = 0$, unless $p = 3, m = 1$ or $p = 2, m \le 2$.

Under assumption $\beta = \beta' = 0$ we have $a_{rs} = 0$ whenever $r, s > 0$. Proceeding as above we conclude $\tilde{\psi} = 0$ unless $p^m \le 2$, i.e., $p = 2, m = 1$. Under assumption $\beta' = 0$ we have $a_{rs} = 0$ whenever $r > 1, s > 0$. Here $\tilde{\psi} = 0$ unless $p^m \le 3$, i.e., $p = 2$ or $3, m = 1$.

These considerations prove our assertion concerning ψ. It remains only to investigate more thoroughly the case $p = 3, n = 2, m = 1$. We have proved already that $\psi(fe_1, ge_1) = \psi(fe_2, ge_2) = 0$. By the above

$$\psi(fe_1, ge_2) = \sum_{0 \le r, s \le 2} a_{rs} \partial^r(f) \partial^s(g),$$

where $a_{rs} = 0$ whenever $r > 1, s > 0$ or, by symmetry, $r > 0, s > 1$. In other words, $a_{rs} = 0$ whenever $r + s > 2$. Put $a := a_{02}$ and

$$\tilde{\psi}(f, g) := \psi(fe_1, ge_2) - \partial^2(afg) = \sum_{0 \le r, s \le 2} a'_{rs} \partial^r(f) \partial^s(g).$$

Then $a'_{rs} = 0$ whenever $r + s > 2$, and also $a'_{02} = 0$. Note that $\partial^2(afg) \in k$ for all $f, g \in B$ since $\partial^3 = 0$. Examining successively coefficients in $\partial(\tilde{\psi}(f, g)) = 0$, we find $a'_{rs} = 0$ for all r, s. Thus $\tilde{\psi} = 0$, and $\psi(fe_1, ge_2) = \partial^2(afg)$. Take in $(*)$ $i = j = 1, k = 2$, and look at the coefficients of e_1. We get

$$\partial^2(agh)\partial(f) - \partial^2(afh)\partial(g) + \gamma(f, g)h + \delta_1(g, h)f - \delta_1(f, h)g = 0.$$

Here $\delta_1(f, h)$ is the coefficient of e_1 in $\varphi'(fe_1, he_2)(e_1)$, while γ does not interest us. Inserting $g = 1$, we find

$$\delta_1(f, h) = \partial^2(ah)\partial(f) + \gamma(f, 1)h + \delta_1(1, h)f.$$

We can write

$$\delta_1(f, h) - \delta_1(1, fh) = \sum_{0 \le r, s \le 2} b_{rs}\partial^r(f)\partial^s(h)$$

with some $b_{rs} \in B$. The expansion of $\delta_1(1, fh)$ is a B-linear combination of fh, $\partial(fh)$, $\partial^2(fh)$, and so contains only terms with $r + s \le 2$. Hence $b_{12} = a, b_{21} = 0$. Let $\delta_2(f, h)$ denote the coefficient of e_2 in $\varphi'(fe_1, he_2)(e_2)$. By symmetry,

$$\delta_2(f, h) - \delta_2(1, fh) = \sum_{0 \le r, s \le 2} c_{rs}\partial^r(f)\partial^s(h),$$

where $c_{rs} \in B, c_{12} = 0$. Now notice that $\psi(fe_1, he_2) = \psi(e_1, fhe_2)$. Hence $\varphi(fe_1, he_2) - \varphi(e_1, fhe_2) \in G'_0$, and $\varphi'(fe_1, he_2) - \varphi'(e_1, fhe_2) \in \rho(G'_0)$. We have

$$\begin{aligned} &\mathrm{tr}_B\big(\varphi'(fe_1, he_2) - \varphi'(e_1, fhe_2)\big) \\ &\quad = \delta_1(f, h) - \delta_1(1, fh) + \delta_2(f, h) - \delta_2(1, fh) \\ &\quad = \sum_{0 \le r, s \le 2} (b_{rs} + c_{rs})\partial^r(f)\partial^s(h). \end{aligned}$$

If $\rho(G'_0) \subset \mathrm{sl}_B(G_{-1})$ then we deduce $b_{rs} + c_{rs} = 0$ for all r, s. In particular, $a = b_{12} + c_{12} = 0$. Hence $\psi = 0$.

PROPOSITION 2.7. *Let $p = 3$. Suppose that B is a G_0-invariant commutative subalgebra of the associative algebra* $\mathrm{End}\, G_{-1}$ *such that B is G_0-simple, G_0 acts nilpotently on B, and G_{-1} is free of rank 2 over B. Assume in addition that $\mathrm{sl}_B(G_{-1}) \subset \rho(G_0)$, where ρ is the representation of G_0 in G_{-1}. Then $H^0(G_0, H^{2,-1}(G)) = 0$.*

Proof. Let e_1, e_2 be a basis for G_{-1} over B, and let E_{ij}, $1 \le i, j \le 2$, be B-linear endomorphisms of G_{-1} defined by the rule $E_{ij} e_l = \delta_{jl} e_i$, $l = 1, 2$. then Be_1, Be_2 are two weight spaces with respect to a one-dimensional torus $T := k(E_{11} - E_{22})$. Suppose that $\overline{\varphi} \in H^{2,-1}(G)$ is annihilated by G_0. We can find its T-invariant representative $\varphi \in Z^{2,-1}(G)$. Then

$$\varphi(fe_1, ge_2) = 0, \qquad \varphi(fe_1, ge_1) = \xi(f, g)e_2,$$
$$\varphi(fe_2, ge_2) = \eta(f, g)e_1$$

for all $f, g \in B$, where ξ, η are certain skewsymmetric bilinear mappings $B \times B \to B$. We have $(E_{12}\varphi)(fe_1, ge_2) = -\xi(f, g)e_2$. On the other hand, $E_{12}\varphi$ is a coboundary, say $d\gamma$, where $\gamma : G_{-1} \to G_0$ is a linear mapping. We have $d\gamma(fe_1, ge_2) = [fe_1, \gamma(ge_2)] + [\gamma(fe_1), ge_2]$. Comparing the coefficients of e_2, we get

$$\xi(f, g) = g\alpha(f) + f\alpha'(g) + \delta(f)(g),$$

where $\alpha, \alpha' : B \to B$ and $\delta : B \to \operatorname{Der} B$ are certain linear mappings. Note that $\delta(B) \subset \pi(G_0)$ where $\pi : G_0 \to \operatorname{Der} B$ is the canonical homomorphism. Put $D := \delta(1) \in \operatorname{Der} B$ and $\delta_0(f) = \delta(f) - fD$. The skewsymmetricity of ξ is written as

$$f(\alpha(g) + \alpha'(g)) + g(\alpha(f) + \alpha'(f)) + \delta(f)(g) + \delta(g)(f) = 0.$$

Inserting $f = g = 1$, we get $\alpha(1) + \alpha'(1) = 0$. Inserting next $g = 1$, we get $\alpha(f) + \alpha'(f) = -D(f)$. It follows that $\delta_0(f)(g) + \delta_0(g)(f) = 0$ for all $f, g \in B$.

Denote by R the set of all linear mappings $\varepsilon : B \to \pi(G_0)$ such that the mapping $\varepsilon_0 : B \to \operatorname{Der} B$ defined by the rule $\varepsilon_0(f) := \varepsilon(f) - f\varepsilon(1)$, $f \in B$, satisfies $\varepsilon_0(f)(g) + \varepsilon_0(g)(f) = 0$ for all $f, g \in B$ or, equivalently, $\epsilon_0(g)(g) = 0$ for all $g \in B$. Note that $\varepsilon_0(gh) = g\varepsilon_0(h) + h\varepsilon_0(g)$ for any such an ε and $g, h \in B$. If $\varepsilon \in R$ and $D' \in \pi(G_0)$ then

$$(D'\varepsilon)(f) := [D', \varepsilon(f)] - \varepsilon(D'f) = f[D', \varepsilon(1)] + (D'\varepsilon_0)(f).$$

Obviously, $(D'\varepsilon)(f) \in \pi(G_0)$ and $D'\varepsilon_0$ satisfies $(D'\varepsilon_0)(g)(g) = 0$. We conclude that $D'\varepsilon \in R$. It follows that the ideal I of B generated by all elements $\varepsilon_0(f)(g)$ with $\varepsilon \in R$ and $f, g \in B$ is G_0-invariant. Hence $I = B$ or $I = 0$. Suppose $I = B$. Let $\mathfrak{m}$ be the maximal ideal of B. There exists ε, f, g such that $u := \varepsilon_0(f)(g) \notin \mathfrak{m}$. Since $B/\mathfrak{m} \cong k$, we have $u = \lambda \pmod{\mathfrak{m}}$ for some nonzero $\lambda \in k$. Subtracting suitable scalars, we may assume $f, g \in \mathfrak{m}$. Note that $g \notin \mathfrak{m}^2$ since $\varepsilon_0(f)(\mathfrak{m}^2) \subset \mathfrak{m}$. The derivation $\varepsilon(fg) = fg\varepsilon(1) + f\varepsilon_0(g) + g\varepsilon_0(f)$ leaves $\mathfrak{m}$, hence also $\mathfrak{m}^2$, invariant.

However, $\varepsilon(fg)(g) = g\varepsilon(f)(g) \equiv \lambda g \pmod{\mathfrak{m}^2}$. It follows that $\varepsilon(fg)$ induces a nonnilpotent linear transformation of $\mathfrak{m}/\mathfrak{m}^2$. Hence $\varepsilon(fg)$ is not nilpotent, a contradiction. Thus $I = 0$, i.e., $\varepsilon_0 = 0$ for every $\varepsilon \in R$. This means that each $\varepsilon \in R$ is a B-linear mapping. As we have seen in the proof of Proposition 2.6, $\pi(G_0)$ contains no nonzero B-submodules. It follows $R = 0$.

We have proved that $\delta \in R$. Hence $\delta = 0$. The skewsymmetricity of ξ now yields $\alpha' = -\alpha$, i.e., $\xi(f, g) = g\alpha(f) - f\alpha(g)$. By symmetry $\eta(f, g) = g\beta(f) - f\beta(g)$ where $\beta : B \to B$ is a linear mapping. Define a linear mapping $\psi : G_{-1} \to G_0$ so that $(\rho \circ \psi)(fe_1) = \alpha(f)E_{21}$ and $(\rho \circ \psi)(fe_2) = \beta(f)E_{12}$ for $f \in B$. We see that $\varphi = d\psi$ is a coboundary.

3. A RECOGNITION CRITERION FOR CARTAN TYPE ALGEBRAS

Most of the basic results describing behavior of Cartan type Lie algebras have counterexamples in low characteristics. This is true, in particular, in regard to filtered deformations. We will show here that even in low characteristics a filtered deformation of a graded Cartan type Lie algebra is itself of Cartan type provided that the filtered and the graded algebras have the same parameters of the minimal embedding into a Witt type Lie algebra.

Let L be a Lie algebra and L_0 its subalgebra. Denote by $P(L)$ the universal p-envelope of L, i.e., the restricted Lie subalgebra generated by L in the universal enveloping algebra $U(L)$. For $r \geq 0$ denote by $P(L)_r$ the subspace of $P(L)$ spanned by the elements D^{p^j} with $D \in L$, $j \leq r$. The height function v on L associated with L_0 is defined as follows [13]: If $D \in L$ then $v(D)$ is the smallest integer $r \geq 0$ such that $D^{p^r} \in P(L)_{r-1} + N_{P(L)}(L_0)$ where $N_{P(L)}(L_0)$ denotes the normalizer of L_0 in $P(L)$ (we set formally $P(L)_{-1} = 0$). Obviously, the value $v(D)$ depends only on the coset of D modulo L_0. Assume that $N_L(L_0) = L_0$. Denote by $\mathbf{m}(L, L_0)$ the n-tuple $(m_1, \dots, m_n)$ such that $n = \dim L/L_0$ and each integer $r > 0$ is repeated among $m_1, \dots, m_n$ the number of times equal to $\dim \mathscr{E}_r/\mathscr{E}_{r-1}$, where $\mathscr{E}_r := \{D \in L | v(D) \leq r\}$.

THEOREM 3.1 [13, Theorem 1.1]. *Suppose that L_0 is selfnormalizing in L and contains no nonzero ideals of L. Let $\mathbf{m}'$ be a n-tuple of positive integers, $n := \dim L/L_0$. There exists a transitive embedding $\tau : L \to W_n(\mathbf{m}')$ such that $L_0 = \tau^{-1}(W_n(\mathbf{m}')_0)$ if and only if $\mathbf{m}' \geq \mathbf{m}(L, L_0)$, where $W_n(\mathbf{m}')_0$ is the distinguished maximal subalgebra of $W_n(\mathbf{m}')$.*

THEOREM 3.2 [18, I, Theorem 8.2]. *Let L be a transitive subalgebra of $W_n(\mathbf{m})$ and $L_0 := L \cap W_n(\mathbf{m})_{\mathbf{0}}$. Denote by Γ the representation of L_0 in*

L/L_0. *Suppose that one of the following conditions holds*:

(1) $\Gamma(L_0) \subset \mathrm{sl}(L/L_0)$,

(2) $L_0 + [L, L] = L$ *and* $\Gamma(L_0 \cap [L, L]) \subset \mathrm{sl}(L/L_0)$,

(3) $\mathrm{sp}'(L/L_0) \subset \Gamma(L_0) \subset \mathrm{sp}(L/L_0)$,

(4) $\mathrm{sp}'(L/L_0) \subset \Gamma(L_0)$ *and* $\Gamma(L_0 \cap [L, L]) \subset \mathrm{sp}(L/L_0)$,

(5) *the subspace* $L_{-1}/L_0 := \Gamma([L_0, L_0])(L/L_0)$ *is of codimension* 1 *in* L/L_0 *and the bilinear mapping* $L_{-1}/L_0 \times L_{-1}/L_0 \to L/L_{-1}$ *induced by the multiplication in* L *is nondegenerate*.

Then there exists a volume, hamiltonian, or contact form ω *on* $W_n(\mathbf{m})$ *such that* L *is contained in* $S_n(\mathbf{m}, \omega)$, $CS_n(\mathbf{m}, \omega)$, $H_n(\mathbf{m}, \omega)$, $CH_n(\mathbf{m}, \omega)$, *or* $K_n(\mathbf{m}, \omega)$ *in the respective cases. The form* ω *is unique up to a multiple.*

We denote by $\mathrm{sp}'(V)$ the commutant of the symplectic Lie algebra $\mathrm{sp}(V)$, which is distinct from $\mathrm{sp}(V)$ in case of $p = 2$. Conditions (1)–(5) correspond to (8.1)–(8.5) of [18, I]. We have replaced (8.4) however, with a slightly weaker condition. The reduction of this case to (3) still can be carried out. Note that $\mathrm{sp}'(V)$ is an irreducible Lie algebra of linear transformations unless $p = 2$, $\dim V = 2$, in which case $\mathrm{sp}'(V)$ consists of scalar transformations. If L satisfies (4), we deduce $[L_0, L] + L_0 = L$, hence also $[L, L] + L_0 = L$. Consider $L' := [L, L] + L'_0$ where $L'_0 := \Gamma^{-1}(\mathrm{sp}(L/L_0))$. Then L' is a transitive subalgebra of L such that $L' \cap L_0 = L'_0$ and $\mathrm{sp}'(L/L_0) \subset \Gamma(L'_0) \subset \mathrm{sp}(L/L_0)$. Hence there exists a hamiltonian form ω (uniquely determined up to a scalar multiple) with the property $L' \subset H_n(\mathbf{m}, \omega)$. Since L' is an ideal of L, for each $D \in L$ the differential form $D\omega$ is annihilated by L', hence is a scalar multiple of ω. It follows $L \subset CH_n(\mathbf{m}, \omega)$.

Following [25], we call a filtration L_i, $i \in \mathbb{Z}$, in a Lie algebra L standard if $L_{i-1} = [L_{-1}, L_i] + L_i$ for all $i \leq -1$ and $L_{i+1} = \{x \in L_i \| [L_{-1}, x] \subset L_i\}$ for all $i \geq 0$, or, in other words, the associated graded algebra G satisfies (g1) and (g2). Every pair $L_0 \subset L_{-1}$ of a subalgebra L_0 and its module L_{-1} determine a standard filtration. A standard filtration is called noncontractible if G satisfies (g3) as well.

PROPOSITION 3.3. *Let* L *be a Lie algebra endowed with an exhaustive separating noncontractible standard filtration. Let* $G := \mathrm{gr}\, L$. *Suppose that* $A(G)_+ \neq 0$ *and that the factor algebra* $G_0/A(G)_0$ *is solvable. Let* $M \subset L$ *be a subalgebra. Then* $\mathrm{gr}\, M \supset A(G)$ *if and only if* $M \supset L^{(\infty)}$, *where* $L^{(\infty)}$ *denotes the final term of the derived series for* L.

Proof. If M satisfies $\mathrm{gr}\, M \supset A(G)$ then $\mathrm{gr}[M, M] \supset [A(G), A(G)] = A(G)$, and it follows $\mathrm{gr}\, M^{(\infty)} \supset A(G)$. In particular, $\mathrm{gr}\, L^{(\infty)} \supset A(G)$. One implication of this proposition is immediate now.

Suppose $\operatorname{gr} M \supset A(G)$. We note first that M is generated by M_{-1}. Indeed, let N be the subalgebra of M generated by M_{-1}. Then $\operatorname{gr}_i N = \operatorname{gr}_i M$ for all $i \geq -1$. In particular, $G_{-1} = A(G)_{-1} = \operatorname{gr}_{-1} M = \operatorname{gr}_{-1} N$. By (g2) $G_- \subset \operatorname{gr} N$. Hence $\operatorname{gr} N = \operatorname{gr} M$, and $N = M$. We see also that $G_- \subset \operatorname{gr} M$, whence $M + L_0 = L$. Next we claim that $M + L_i$ is a subalgebra for any i. We may assume that $i > 0$, in which case L_i is a subalgebra. Let $\pi : L_{i-1} \to G_{i-1}$ denote the canonical projection. Since $\pi([M_{-1}, L_i]) = [G_{-1}, G_i] \subset A(G)_{i-1} \subset \operatorname{gr}_{i-1}\mathbf{M}$,we get $[M_{-1}, L_i] \subset \pi^{-1}(\operatorname{gr}_{i-1} M) \subset M + L_i$. Hence M_{-1} normalizes $M + L_i$. Being generated by M_{-1}, the algebra M normalizes $M + L_i$ as well. Thus $[M, L_i] \subset M + L_i$, as required.

Denote by σ the representation of M in L/M. The subspaces $(L_i + M)/M$ form an exhaustive separating filtration of L/M. By the above it consists of submodules with respect to σ. Denote by J the kernel of the representation of M in the associated graded module $\operatorname{gr} L/M$. Then J is an ideal of M containing $M_1 := M \cap L_1$. Hence $\operatorname{gr} J$ is an ideal of $\operatorname{gr} M$ containing $\operatorname{gr}_1 M$, and in particular $A(G)_1$. By Lemma 1.3 the ideal of $A(G)$ generated by $A(G)_1$ is equal to $A(G)$. It follows $A(G) \subset \operatorname{gr} J$. In particular, $G_- \subset J$. Hence $M = M_0 + J$. Furthermore, $M/J \cong M_0/J \cap M_0 \cong \operatorname{gr}_0 M/\operatorname{gr}_0 J$. Since $\operatorname{gr}_0 J \supset A(G)_0$, this factor algebra is solvable. Obviously, J acts nilpotently in L/M, whence $\sigma(J)$ is nilpotent. Since $\ker \sigma \subset J$, we have $\sigma(M)/\sigma(J) \cong M/J$. We conclude that $\sigma(M)$ is solvable.

In proving the inclusion $M \supset L^{(\infty)}$ we may assume that M is a minimal subalgebra satisfying $\operatorname{gr} M \supset A(G)$. Then $[M, M] = M$. We deduce that $\sigma(M) = 0$, i.e., M is an ideal of L. Note that $L_0/(M_0 + L_1) \cong G_0/\operatorname{gr}_0 M$ is solvable. Since L_1 is nilpotent, $L/M \cong L_0/M_0$ is solvable as well, whence the required inclusion.

THEOREM 3.4. *Let G be a graded Lie algebra of Cartan type related to an n-tuple $\mathbf{m}$, and let L be a filtered deformation of G. If $p = 2$, $n = 2$ $m_2 = 1$, and G is hamiltonian, assume that $G_0 \cong \mathrm{sl}(G_{-1})$ and $G_1 \neq 0$. Then L is a filtered Lie algebra of Cartan type related to $\mathbf{m}$ whenever the equality $\mathbf{m}(L, L_0) = \mathbf{m}$ holds. More precisely, there exists a volume, hamiltonian, or contact form ω on $W_n(\mathbf{m})$ such that*

(1) *if $S_n'(\mathbf{m}) \subset G \subset S_n(\mathbf{m})$ then $S_n'(\mathbf{m}, \omega) \subset L \subset S_n(\mathbf{m}, \omega)$,*

(2) *if $S_n'(\mathbf{m}) \subset G \subset CS_n(\mathbf{m})$ then $S_n'(\mathbf{m}, \omega) \subset L \subset CS_n(\mathbf{m}, \omega)$,*

(3) *if $H_n''(\mathbf{m}) \subset G \subset H_n(\mathbf{m})$ then $H_n''(\mathbf{m}, \omega) \subset L \subset H_n(\mathbf{m}, \omega)$,*

(4) *if $H_n''(\mathbf{m}) \subset G \subset CH_n(\mathbf{m})$ then $H_n''(\mathbf{m}, \omega) \subset L \subset CH_n(\mathbf{m}, \omega)$,*

(5) *if $K_n'(\mathbf{m}) \subset G \subset K_n(\mathbf{m})$ then $K_n'(\mathbf{m}, \omega) \subset L \subset K_n(\mathbf{m}, \omega)$.*

Proof. Let $\tau : L \to W_n(\mathbf{m})$ be a minimal embedding afforded by Theorem 3.1. If G satisfies the hypotheses of (1), (3), or (5) then the respective condition of Theorem 3.2 holds. Suppose that G satisfies the hypotheses of either (2) or (4). We identify G_0 with the Lie algebra $\Gamma(L_0) \subset \mathrm{gl}(G_{-1})$. Suppose that $\Gamma(L_0 \cap [L, L]) = \mathrm{gr}_0[L, L] \not\subset G'_0$ where $G'_0 := \mathrm{sl}(G_{-1})$ or $G_0 \cap \mathrm{sp}(G_{-1})$, respectively. Then $G_0 = \mathrm{gl}(G_{-1})$ in the first case, while $\mathrm{sp}'(G_{-1}, b) \subset G_0 \subset \mathrm{csp}(G_{-1}, b), G_0 \not\subset \mathrm{sp}(G_{-1}, b)$ in the second, where we use the notation b to specify a skewsymmetric nondegenerate form on G_{-1}. If $p > 2$ then actually $G_0 = \mathrm{csp}(G_{-1}, b)$ in the second case. In all cases G_0 has a nontrivial center (if $p = 2$ then $\mathrm{sp}'(G_{-1}, b)$ contains the identity transformation of G_{-1}). Note that $G'_0 \supset [G_{-1}, G_1] + [G_0, G_0]$. By Lemma 2.3 and Theorem 2.1(2), $\mathrm{gr}_0[L, L_0] \subset G'_0$. Since $\dim G_0/G'_0 = 1$, we have $G'_0 + \mathrm{gr}_0[L, L] = G_0$. By Lemma 2.2 there exists a nonzero G_0-invariant bilinear form on G_{-1}. This immediately leads to a contradiction when $G_0 = \mathrm{gl}(G_{-1})$. Any bilinear form on G_{-1} which is invariant with respect to $\mathrm{sp}'(G_{-1}, b)$ is a scalar multiple of b. However, b is not G_0-invariant, again a contradiction. Thus either (2) or (4) of Theorem 3.2 does hold.

We conclude $\tau(L) \subset \tilde{L}$ where $\tilde{L} := S_n(\mathbf{m}, \omega)$, $CS_n(\mathbf{m}, \omega)$, $H_n(\mathbf{m}, \omega)$, $CH_n(\mathbf{m}, \omega)$, or $K_n(\mathbf{m}, \omega)$ in the respective cases. The associated graded algebra $\tilde{G} := \mathrm{gr}\, \tilde{L}$ is of the same Cartan type as G. Since $L_0 = \tau^{-1}(\tilde{L}_0)$, the filtration of L is induced by the canonical filtration of $\tilde{L}$. Hence τ induces an embedding of graded algebras $\tau' : G \to \tilde{G}$. Both $A(G)$ and $A(\tilde{G})$ are graded Lie algebras of Cartan type isomorphic to $S'_n(\mathbf{m})$, $H''_n(\mathbf{m})$, or $K'_n(\mathbf{m})$ in the respective cases. Since τ' maps $A(G)$ into $A(\tilde{G})$, we must have $\tau'(A(G)) = A(\tilde{G})$. Thus $\mathrm{gr}\, \tau(L) = \tau'(G) \supset A(\tilde{G})$. Straightforward examining of Cartan type shows that $\tilde{G}_0/A(\tilde{G})_0$ is solvable and $A(\tilde{G})_+ \neq 0$. By Proposition 3.3, $\tau(L) \supset \tilde{L}^{(\infty)}$. Thus $\tau(L)$ is of Cartan type.

Convenient conditions ensuring the equality $\mathbf{m}(L, L_0) = \mathbf{m}(G, G_0 + G_+)$ are provided by [13, Theorem 3.2]. The equality is fulfilled for Cartan type Lie algebras of characteristic $p > 3$ [30, Proposition 5.1]. The cases $p = 2, 3$ require a more careful analysis. Here we will need a result on equality of parameters of the minimal embedding in a form given by H. Strade.

If G is a graded Lie algebra then its derivation algebra $\mathrm{Der}\, G$ carries induced gradation. Put $\mathrm{Der}_- G = \bigoplus_{i<0} \mathrm{Der}_i G$. Denote by $\mathrm{Der}'_- G$ the restricted subalgebra of $\mathrm{Der}_- G$ generated by $\mathrm{ad}\, G_-$. Then $\mathrm{Der}'_- G$ is a homogeneous ideal of $\mathrm{Der}_- G$. We put $\mathfrak{D}(G) = \mathrm{Der}_- G/\mathrm{Der}'_- G$ and $\mathfrak{D}_i(G) = \mathrm{Der}_i G/\mathrm{Der}'_i G$ for $i < 0$.

THEOREM 3.5 [25]. *Let $L = L_{-r} \supset L_{-r+1} \supset \dots$ be a standard filtration in a Lie algebra L and $G = \mathrm{gr}\, L$. Assume that $r < p$ and $G_0 + G_+$ is selfnormalizing in G. If $\mathfrak{D}_i(G) = 0$ for all $i < 0$, i.e., if $\mathrm{Der}_- G$ is generated as a restricted algebra by $\mathrm{ad}\, G_-$, then $\mathbf{m}(L, L_0) = \mathbf{m}(G, G_0 + G_+)$.*

PROPOSITION 3.6. *Let G be a graded Lie algebra of depth $r > 0$. Suppose that G_{-r} is an irreducible G_0-module and the centralizer of G_- in G coincides with G_{-r} (these assumptions are automatically fulfilled whenever G satisfies* (g1)–(g4)). *Put $T_i := \{x \in G_i | Dx = 0$ for all $D \in \mathrm{Der}'_{-i-r}G\}$ and $T'_i := \sum_{0<j<i}[G_j, G_{i-j}]$. Then $T_i \supset T'_i$ and there are canonical embeddings $\mathfrak{D}_{-i-r}(G) \to \mathrm{Hom}_{G_0}(T_i/T'_i, G_{-r})$ for $i > 0$. If $r < p$ then there is also an embedding $\mathfrak{D}_{-r}(G) \to H^1(G_0, G_{-r})$.*

Proof. Let $i \geq 0$. We first establish the injectivity of the mapping $\alpha : \mathrm{Der}_{-i-r}G \to \mathrm{Hom}(G_i, G_{-r})$ obtained by restricting derivations to G_i. Suppose that $D \in \mathrm{Der}_{-i-r}G$ and $D(G_i) = 0$. Then $D(G) \subset \bigoplus_{j>-r} G_j$. Furthermore, since $D(G_-) = 0$, we have $[G_-, D(G)] = D([G_-, G]) \subset D(G)$. Since G_- acts nilpotently on G, every nonzero G_--invariant subspace of G contains a nonzero element annihilated by G_-. By assumptions on G, it follows $D(G) = 0$.

If $i = 0$ then the image of α is contained in the group of 1-cocycles $Z^1(G_0, G_{-r})$. In this case $\alpha(D)$ is a coboundary if and only if $D \in \mathrm{ad}\, G_{-r}$. Hence the kernel of the mapping $\mathrm{Der}_{-r}G \to H^1(G_0, G_{-r})$ induced by α coincides with $\mathrm{ad}\, G_{-r}$. Notice that $\mathrm{Der}'_{-r}G = \mathrm{ad}\, G_{-r}$ provided $r < p$.

Assume that $i > 0$. The image of α is contained in $\mathrm{Hom}_{G_0}(G_i, G_{-r})$ because $D(G_0) = 0$ for every $D \in \mathrm{Der}_{-i-r}G$. If $D \in \mathrm{Der}_{-i-r}G$ then $D(T'_i) \subset D([G_+, G_+]) \subset [G_+, G] \subset \bigoplus_{j>-r} G_j$, whence $D(T'_i) = 0$. In particular, $T'_i \subset T_i$ and α induces a mapping $\beta : \mathrm{Der}_{-r-i}G \to \mathrm{Hom}_{G_0}(T_i/T'_i, G_{-r})$. Obviously $\ker \beta \supset \mathrm{Der}'_{-r-i}G$. It remains only to verify that the equality holds here. Note that α induces a mapping $\gamma : \ker \beta \to \mathrm{Hom}_{G_0}(G_i/T_i, G_{-r})$ which is injective since α is. We will show that the restriction of γ to $\mathrm{Der}'_{-r-i}G$ is surjective. The claim then follows.

Note that $[\mathrm{Der}'_{-r-i}G, \mathrm{ad}\, G_0] \subset \mathrm{ad}\, G_{-r-i} = 0$, i.e., $E := \mathrm{Der}'_{-r-i}G$ is a trivial G_0-module. The dual vector space E^* is also a trivial G_0-module. Consider the mapping $\theta : G_i/T_i \to \mathrm{Hom}(E, G_{-r}) \cong E^* \otimes G_{-r}$ defined by the rule $\theta(x + T_i)(D) = \gamma(D)(x + T_i) = D(x)$ for $x \in G_i, D \in E$. Obviously, θ is a homomorphism of G_0-modules. It is injective according to the definition of T_i. The G_0-module $E^* \otimes G_{-r}$ is a sum of submodules isomorphic to G_{-r}, hence is completely reducible. It follows that the transpose of θ which is non other than $\gamma : E \cong \mathrm{Hom}_{G_0}(E^* \otimes G_{-r}, G_{-r}) \to \mathrm{Hom}_{G_0}(G_i/T_i, G_{-r})$ is surjective.

LEMMA 3.7. *Let $\tilde{G}$ be a graded Lie algebra of depth r and G its homogeneous subalgebra. Suppose that the centralizer of G_- in $\tilde{G}$ coincides with $\tilde{G}_{-r}$. Denote by $\tilde{Z}^1$ the group of homogeneous* 1*-cocycles $G \to \tilde{G}$ of degree $-r$, by $\tilde{H}^1$ its factor group modulo coboundaries. There are canonical embeddings $\tilde{Z}^1 \to Z^1(G_0, \tilde{G}_{-r})$ and $\tilde{H}^1 \to H^1(G_0, \tilde{G}_{-r})$. Furthermore, a homogeneous linear mapping $\varphi : G \to \tilde{G}$ of degree $-r$ is a* 1*-cocycle if and* *only if* φ *is a* G_-*-module homomorphism and its restriction* $G_0 \to \tilde{G}_{-r}$ *is a* 1-*cocycle*.

Proof. Let φ be homogeneous of degree $-r$. The condition that φ is a cocycle means $\varphi([x, y]) = [x, \varphi(y)] + [\varphi(x), y]$ for all $x, y \in G$. Assume $x \in G_i$, $y \in G_j$. If $i < 0$, then $\varphi(x) = 0$ and the equality means that φ commutes with $\operatorname{ad} x$. For $i = j = 0$ it is the cocyle condition for the restriction $G_0 \to \tilde{G}_{-r}$. Suppose that the equality does hold for those values of i, j. We claim that it holds then for all i, j. We proceed by induction on $i + j$ and may assume $i + j > 0$. Then both sides of the equality are in $\tilde{G}_l$ where $l = i + j - r > -r$. Since for all $z \in G_-$

$$\begin{aligned}
&[z, \varphi([x, y]) - [x, \varphi(y)] - [\varphi(x), y]] \\
&\quad = \varphi([[z, x], y]) - [[z, x], \varphi(y)] - [\varphi([z, x]), y] \\
&\qquad + \varphi([x, [z, y]]) - [x, \varphi([z, y])] - [\varphi(x), [z, y]] = 0,
\end{aligned}$$

our claim follows. If $\varphi(G_0) = 0$ then $\varphi(G)$ is a G_--invariant subspace contained in $\bigoplus_{l > -r} \tilde{G}_l$. We then get $\varphi(G) = 0$, i.e., the canonical mapping $\tilde{Z}^1 \to Z^1(G_0, \tilde{G}_{-r})$ is injective. Since the coboundaries are in each case determined by the elements of $\tilde{G}_{-r}$, the induced mapping of cohomology groups is injective as well.

4. FILTERED DEFORMATIONS OF HAMILTONIAN ALGEBRAS

We shall apply the results of the preceding section to describe filtered deformations of hamiltonian Lie algebras G, $H_2''(\mathbf{m}) \subset G \subset H_2(\mathbf{m})$. The case $p = 3$, $m_2 = 1$ presents the main difficulty. It will be useful to introduce special notations for certain graded hamiltonian Lie algebras of characteristic 3:

$$\begin{aligned}
X'(m_1) &:= H_2''(m_1, 1), \\
X(m_1) &:= X'(m_1) + k\mathscr{D}\left(x_1^{(p^{m_1}-1)} x_2^{(2)}\right) + k\mathscr{D}\left(x_1^{(p^{m_1})}\right).
\end{aligned}$$

THEOREM 4.1. *Let* $L = L_{-1} \supset L_0 \supset \dots$ *be a standard separating filtration in a Lie algebra* L, *and let* $G := \operatorname{gr} L$ *be its associated graded algebra. Assume that* $\dim G_{-1} = 2$, $G_0 \cong \mathrm{sl}(G_{-1})$, $G_1 \neq 0$. *Then* L *is isomorphic as an abstract Lie algebra to a hamiltonian Lie algebra* $\tilde{L}$, $H_2''(\mathbf{m}, \omega) \subset \tilde{L} \subset H_2(\mathbf{m}, \omega)$. *Furthermore*, L *is isomorphic to a hamiltonian Lie algebra* $\tilde{L}$ *as a filtered Lie algebra unless* $p = 3, m_2 = 1, G = X(m_1)$, *or* $G = X'(m_1) + k\mathscr{D}(x_1^{(p^{m_1})})$. *The graded Lie algebra* $H_2''(\mathbf{m})$ *has no nontrivial filtered deformations*.

The proof will be given at the end of this section. We shall use the notations $\mathfrak{D}(G)$, T_i, T_i' introduced in Section 3. Put $T := \bigoplus_{i \geq 1} T_i$ and $T' := \bigoplus_{i \geq 1} T_i' = [G_+, G_+]$. When we need to specify the graded Lie algebra we write $T(G)$ and $T'(G)$ instead.

LEMMA 4.2. *Let $G = H_2''(\mathbf{m})$. The cosets of the elements $\mathscr{D}(x_i^{(p^s)} x_j)$, $1 \leq i, j \leq 2, 0 < s < m_i$, constitute a basis for G_+/T. The cosets of the elements $\mathscr{D}(x_i^{(p^s)})$, $1 \leq i \leq 2, s < m_i, p^s \geq 4$, and in addition the elements $\mathscr{D}(x_1^{(p^s-1)} x_2^{(p-1)})$ with $0 < s < m_1$ if $p = 3, m_2 = 1$ or with $1 < s < m_1$ if $p = 2, m_2 = 1$ constitute a basis for $(T \cap G_{(2)})/T'$ where $G_{(2)} = \bigoplus_{l \geq 2} G_l$. The algebra G is generated by its components G_{-1}, G_0, G_1, and the elements $\mathscr{D}(x_i^{(p^{m_i-1})} x_j)$, $1 \leq i, j \leq 2$.*

Proof. As is well known, a basis for G consists of the elements $\mathscr{D}(x_1^{(a)} x_2^{(b)})$ with $0 \leq a < p^{m_1}$, $0 \leq b < p^{m_2}$, $(a, b) \neq (p^{m_1} - 1, p^{m_2} - 1)$, and $(a, b) \neq (0, 0)$. Straightforward computations [9, III, Proposition 1] show that the monomials in x_1, x_2 of degree at least 4 and distinct from the ones indicated in the statement of the lemma constitute a basis for T'. Next, $\mathscr{D}(f) \in T$ if and only if $\mathscr{D}(f) \in G_+$ and $\partial_i^{p^s}(f)$ is a linear combination of the monomials $x_1^{(a)} x_2^{(b)}$ with $a + b \neq 1$ for all $i = 1, 2$ and $s \geq 0$. Hence the monomials in x_1, x_2 of degree at least 3 and distinct from $x_i^{(p^s)} x_j$ give a basis for T. One verifies straightforwardly that the subalgebra H generated by G_{-1}, G_0, G_1 and the elements $\mathscr{D}(x_i^{(p^{m_i-1})} x_j)$, $1 \leq i, j \leq 2$, contains also all the elements indicated in the statement of the lemma. Hence H contains a subspace of G_+ complementary to T'. As G_+ is nilpotent, it is generated by this subspace. It follows $H \supset G_+$, and $H = G$.

PROPOSITION 4.3. *Assume $H_2''(\mathbf{m}) \subset G \subset H_2(\mathbf{m})$. Then $\mathscr{D}_i(G) = 0$ for all i except for the following cases*:

(1) $p = 3$, $G = H_2''(\mathbf{1})$, $\dim \mathfrak{D}_{-2}(G) = 1$, $\dim \mathfrak{D}_{-1}(G) = 2$;

(2) $p = 3$, $m_2 = 1$, $G = X'(m_1)$, $X'(m_1) + k\mathscr{D}(x_1^{(p^{m_1})})$ *or* $X(m_1)$, $G \neq H_2''(\mathbf{1})$, $\dim \mathfrak{D}_{-1}(G) = 1$. *In this case there exists a derivation* $\Delta \in \operatorname{Der}_{-1} G \setminus \operatorname{ad} G_{-1}$ *which leaves the subalgebra $\mathscr{D}(kx_1x_2 + kx_1^{(2)}) + G_+$ invariant.*

Proof. Put $G' = H_2''(\mathbf{m})$. One sees from Lemma 4.2 that $\mathscr{D}(x_1x_2)$ annihilates $T(G') \cap G'_{(2)}/T'(G') = \bigoplus_{l>1} T_l(G')/T_l'(G')$. Since the eigenvalues of $\mathscr{D}(x_1x_2)$ on G_{-1} are nonzero, it follows $\operatorname{Hom}_{G_0'}(T_l(G')/T_l'(G'), G_{-1}) = 0$ for any $l > 1$. Note that $[G_0', T(G)] \subset G' \cap T(G) = T(G')$. Thus G_0' annihilates $T(G)/T(G')$, hence also the cokernel C of the canonical mapping $T_l(G')/T_l'(G') \to T_l(G)/T_l'(G)$. It follows $\operatorname{Hom}_{G_0'}(C, G_{-1}) = 0$, whence $\operatorname{Hom}_{G_0'}(T_l(G)/T_l'(G), G_{-1}) = 0$ for $l > 1$. By Proposition 3.6, $\mathscr{D}_{-l}(G) = 0$ for $l > 2$.

Suppose $l = 1$. If $p = 2$ then $T_1 = 0$. Otherwise $T_1 = G_1$. If $p > 3$ then G_1 is irreducible and $\dim G_1 = 4 > \dim G_{-1} = 2$. If $p = 3$ then G_1 contains the smallest nonzero submodule V of dimension 2. According to the representation theory of $G_0 \cong \mathrm{sl}(2)$, we have $V \cong G_{-1}$. The action of G_0 in the factor module G_1/V is trivial. We conclude that $\mathrm{Hom}_{G_0}(T_1/T_1', G_{-1}) = 0$, hence $\mathfrak{D}_{-2}(G) = 0$, unless $p = 3, G_1 = V$. Suppose $p = 3, G_1 = V$. Then either $G_2 = 0$ and $G = H_2''(\mathbf{1})$ or G_2 is spanned by $\mathscr{D}(x_1^{(2)}x_2^{(2)})$ and $G = H_2'(\mathbf{1})$. In the latter case $\mathrm{Der}_{-2}(G) = 0$. Indeed, each $\Delta \in \mathrm{Der}_{-2}(G)$ is a G_0-module endomorphism because $\Delta(G_0) = 0$. Hence $\Delta(G_2)$ is an ideal of G_0 of dimension at most 1. It follows $\Delta(G_2) = 0$. Since $[G_{-1}, G_2] = G_1$, we deduce $\Delta(G_1) = 0$ as well, i.e., $\Delta = 0$. Suppose $G = H''(\mathbf{1})$. Let $\Delta : G \to G$ be an endomorphism of degree -2 such that its restriction $G_1 \to G_{-1}$ is a G_0-module homomorphism. The assignment $u \wedge v \to [\Delta(u), v] + [u, \Delta(v)]$ for $u, v \in G_1$ defines a G_0-module homomorphism $\beta : \wedge^2 G_1 \to G_0$. Since $\dim \wedge^2 G_1 = 1$, it follows $\beta = 0$. It is immediate now that $\Delta \in \mathrm{Der}\, G$. Thus $\mathfrak{D}_{-2}(G) \cong \mathrm{Der}_{-2} G \cong \mathrm{Hom}_{G_0}(G_1, G_{-1})$ has dimension 1.

Consider now $H^1(G_0, G_{-1})$. Since G_0 annihilates the cohomology group, it can be nonzero only if the eigenvalues of $\mathscr{D}(x_1x_2)$ on G_{-1}, i.e., $-1, 1$, and those on G_0, i.e., $-2, 0, 2$ are not all distinct. That happens only when $p = 3$ which we assume from now on. Furthermore, each cohomology class is represented by a cocycle sending $\mathscr{D}(x_1x_2)$ to zero, $\mathscr{D}(x_1^{(2)})$ to a multiple of $\mathscr{D}(x_2)$, and $\mathscr{D}(x_2^{(2)})$ to a multiple of $\mathscr{D}(x_1)$. It is straightforward that any such a mapping is indeed a cocycle. Hence $\dim H^1(G_0, G_{-1}) = 2$. Note that the component of degree -1 in $\tilde{G} := W_2(\mathbf{m})$ coincides with G_{-1} and is equal to its own centralizer in $\tilde{G}$. Using notations of Lemma 3.7, we get embeddings $\mathrm{Der}_{-1} G \subset \tilde{Z}^1 \to Z^1(G_0, G_{-1})$ and $\mathscr{D}_{-1}(G) \to \tilde{H}^1 \to H^1(G_0, G_{-1})$. Define a mapping $\theta : G_{-1} \to \mathrm{Hom}(G, \tilde{G})$ by the rule

$$\theta(\partial)(\mathscr{D}(f)) = \partial^2(f)\partial, \qquad \partial \in G_{-1}, \mathscr{D}(f) \in G,$$

and a skewsymmetric bilinear form on G_{-1} by the rule $\langle \partial_1, \partial_2 \rangle = 1$. If $\partial, \partial' \in G_{-1}$ then $\partial(f)\partial' - \partial'(f)\partial = \langle \partial, \partial' \rangle \mathscr{D}(f)$, and we have

$$\begin{aligned}
&(\partial + \partial')^2(f)(\partial + \partial') - \partial^2(f)\partial - (\partial')^2(f)\partial' \\
&\quad = \big((\partial')^2(f) - \partial\partial'(f)\big)\partial + \big(\partial^2(f) - \partial\partial'(f)\big)\partial' \\
&\quad = [\partial - \partial', \partial(f)\partial' - \partial'(f)\partial] \\
&\quad = \langle \partial, \partial' \rangle [\partial - \partial', \mathscr{D}(f)],
\end{aligned}$$

whence $\theta(\partial+\partial')-\theta(\partial)-\theta(\partial')=\langle\partial,\partial'\rangle\mathrm{ad}(\partial-\partial')$. We have also

$$\theta(\partial)([\partial',\mathscr{D}(f)])=\theta(\partial)(\mathscr{D}(\partial'(f)))=\partial^2\partial'(f)\partial=[\partial',\partial^2(f)\partial],$$

i.e., $\theta(\partial)$ commutes with $\mathrm{ad}\,G_{-1}$. Note that $\theta(\partial_1)$ sends $\mathscr{D}(x_1x_2)$ and $\mathscr{D}(x_2^{(2)})$ to zero, $\mathscr{D}(x_1^{(2)})$ to ∂_1. Similarly, $\theta(\partial_2)$ sends $\mathscr{D}(x_1^{(2)})$ and $\mathscr{D}(x_1x_2)$ to zero, $\mathscr{D}(x_2^{(2)})$ to ∂_2. Thus the restrictions of $\theta(\partial_1)$ and $\theta(\partial_2)$ to G_0 are cocycles. By Lemma 3.7, $\theta(\partial_1),\theta(\partial_2)\in\tilde{Z}^1$. We conclude $\tilde{H}^1\cong H^1(G_0,G_{-1})$. Furthermore, $\theta(\partial)\in\tilde{Z}^1$ for any $\partial\in G_{-1}$ and θ induces a semilinear isomorphism $G_{-1}\cong\tilde{H}^1$.

Suppose that $\Delta\in\mathrm{Der}_{-1}G\setminus\mathrm{ad}\,G_{-1}$. Then $\Delta\in\tilde{Z}^1$ and, subtracting a suitable inner derivation, we may assume $\Delta=\theta(\partial)$ for some $\partial\in G_{-1}$. We deduce that $\partial^2(f)\partial\in G\subset H_2(\mathbf{m})$, hence $\partial^3(f)=0$, whenever $\mathscr{D}(f)\in G$. Assuming $m_1\geq m_2$, we must have $m_2=1$. If $m_1>1$ then necessarily $\partial=\partial_2$. If $\mathbf{m}=\mathbf{1}$ we may adjust generators x_1,x_2 and assume so. It follows $X'(m_1)\subset G\subset X(m_1)$. Note that Δ sends $\mathscr{D}(x_1^{(p^{m_1}-1)}x_2^{(2)})$ to $x_1^{(p^{m_1}-1)}\partial_2=\mathscr{D}(x_1^{(p^{m_1})})$. Since G is stable under Δ, either $G=X'(m_1)$ or G contains $\mathscr{D}(x_1^{(p^{m_1})})$. It is immediate from the explicit formula for Δ that the subalgebra $\mathscr{D}(kx_1x_2+kx_1^{(2)})+G_+$ is stable under Δ. We also see that G is stable under both $\theta(\partial_1)$ and $\theta(\partial_2)$ if and only if $\mathbf{m}=\mathbf{1}$ and $G=H_2''(\mathbf{1})$.

Remark. If G is a graded Lie algebra of Cartan type related to an n-tuple $\mathbf{m}$, it can be proved that $\mathrm{Der}\,G/\mathrm{Der}^{\mathrm{st}}G\cong\mathfrak{D}(G)$ where $\mathrm{Der}^{\mathrm{st}}G$ is the Lie algebra of standard derivations, by which we mean the normalizer of G in the derivation algebra of $O_n(\mathbf{m})$. Consider the case $p=3, G=H_2''(\mathbf{1})$. Here $\mathrm{Der}^{\mathrm{st}}G\cong CH_2(\mathbf{1})$ and computation shows $\dim\mathrm{Der}\,G/\mathrm{ad}\,G=7$. Note that G is isomorphic to the classical Lie algebra of type A_2, which is a simple ideal of the classical Lie algebra of type G_2 generated by "short" roots. Comparing the dimensions, we see that $\mathrm{Der}\,G$ is classical of type G_2.

In what follows we shall deal with two filtrations $(L_i)_{i\in\mathbb{Z}}$ and $(L'_j)_{j\in\mathbb{Z}}$ in a Lie algebra L. For every pair of subspaces $M\supset N$ of L the factor M/N carries two induced filtrations, whose terms are $(M\cap L_i+N)/N$ and $(M\cap L'_j+M)/N$, respectively. Denote by gr_iM/N and gr'_jM/N the factors of these two filtrations in M/N. In particular, the second (respectively the first) filtration in L induces a filtration in gr_iL (respectively gr'_jL). There are canonical isomorphisms

$$\mathrm{gr}_i(\mathrm{gr}'_jL)\cong L_i\cap L'_j/(L_{i+1}\cap L'_j+L_i\cap L'_{j+1})\cong\mathrm{gr}'_j(\mathrm{gr}_iL).$$

PROPOSITION 4.4 ($p=3$). *Let $L=L_{-1}\supset L_0\supset\dots$ be a standard filtration in a Lie algebra L. Suppose that the associated graded algebra G satisfies $X'(m_1)\subset G\subset X(m_1)$. Then L contains a subalgebra L'_0, $L_0\supset L'_0\supset L_1$, and* *an L'_0-invariant subspace $L'_{-1} \supset L_0$ such that the graded algebra G' associated with the corresponding standard filtration enjoys the following properties*:

(1) $G'_{-3} = 0$, $\dim G'_{-2} = \dim G'_0 = 1$, $\dim G'_{-1} = \dim G'_1 = 2$,

(2) *G'_0 acts on G'_{-1} via scalar transformations*,

(3) *the mappings $G'_{-1} \times G'_{-1} \to G'_{-2}$ and $G'_{-1} \times G'_1 \to G'_0$ induced by the multiplication in G' are nondegenerate pairings*.

Proof. Put $U_{-1} := \mathscr{D}(kx_1)$, $U_0 := N_{G_0}(U_{-1}) = \mathscr{D}(kx_1x_2 + kx_1^{(2)})$, and $U_i := G_i$ for $i > 0$. Since $G_1 \subset \mathscr{D}(kx_1^{(3)} + kx_1^{(2)}x_2 + kx_1x_2^{(2)})$, we see that $U := \oplus U_i$ is a homogeneous subalgebra of G. Let $\pi_i : L_i \to G_i$ denote the canonical projections. Put $L'_i := \pi_i^{-1}(U_i)$ for $i = -1, 0$. Then $L'_{-2} := [L'_{-1}, L'_{-1}] + L'_{-1} \supset [L_0, L'_{-1}] + L'_{-1} = L$ since G_{-1} is G_0-irreducible. Hence $\dim G'_{-2} = \dim G_{-1}/U_{-1} = 1$. Next, $\dim G'_{-1} = \dim U_{-1} + \dim G_0/U_0 = 2$. Since $\pi_0([L'_{-1}, L_1]) = [U_{-1}, G_1] \subset U_0$, we have $[L'_{-1}, L_1] \subset L'_0$, i.e., $L_1 \subset L'_1$.

We can derive some information about L from consideration Spencer homology groups. Since $\dim G_{-1} = 2$ and $G_0 \cong \mathrm{sl}(G_{-1})$, a special case of Proposition 2.7 with $B = k$ yields $H^0(G_0, H^{2,-1}(G)) = 0$. Next, $H^{10}(G) \cong \tilde{G}_1/G_1$ where $\tilde{G}_1$ is the first Cartan prolongation of the pair G_{-1}, G_0. Since G_0 annihilates $\tilde{G}_1/G_1$ and $[G_0, G_0] = G_0$, we get $H^1(G_0, H^{10}(G)) = 0$. Let P be the subspace given by Theorem 2.1(1) and $Q := N_{L_0}(P)$.

Put $M := P \cap L'_{-1} + L'_0$. Since $L'_0 = (Q + L_1) \cap L'_0 = Q \cap L'_0 + L_1$ and $P \cap L'_{-1}$ is stable under $Q \cap L'_0$, we get

$$[L'_0, M] \subset [Q \cap L'_0, P \cap L'_{-1}] + [L'_0, L'_0] + [L_1, L'_{-1}] \subset M.$$

Note that $M \cap L_0 = L'_0$ and $M + L_0 = L'_{-1}$. Now $G'_{-1} := L'_{-1}/L'_0 \cong \mathrm{gr}'_{-1}M \oplus \mathrm{gr}'_{-1}L_0$ is a sum of two one-dimensional G'_0-submodules. Furthermore, $\mathrm{gr}'_{-1}M = M/L'_0 = \mathrm{gr}_{-1}M \cong U_{-1}$ and $\mathrm{gr}'_{-1}L_0 = L_0/L'_0 \cong G_0/U_0$. We may regard these as modules over the algebra $U_0 \cong L'_0/L_1$. Since $\mathscr{D}(x_1x_2)$ acts as minus identity transformation in both the modules and $\mathscr{D}(x_1^{(2)})$ annihilates them, $U_{-1} \cong G_0/U_0$. Thus the image of G'_0 in $\mathrm{gl}(G'_{-1})$ consists of scalar transformations. Since G'_{-1} is a faithful G'_0-module, $\dim G'_0 = 1$.

It follows from the above considerations that $L'_1 = \pi_0^{-1}(k\mathscr{D}(x_1^{(2)}))$. Since $[L_0, L_1] \subset L_1 \subset L'_1$, we have $[\mathrm{gr}'_{-1}L_0, \mathrm{gr}'_1 L_1] = 0$. On the other hand, the equality $[x_1, x_1x_2^{(2)}] = x_1x_2$ shows that $[G'_{-1}, \mathrm{gr}'_1 L_1] \neq 0$. Hence the annihilator of $\mathrm{gr}'_1 L_1$ with respect to the pairing $G'_{-1} \times G'_1 \to G'_0$ coincides with $\mathrm{gr}'_{-1}L_0$. As $[x_2^{(2)}, x_1^{(2)}] = -x_1x_2$, we have $[\mathrm{gr}'_{-1}L_0, \mathrm{gr}'_1 L_0] \neq 0$, however. Hence the pairing has zero left kernel. It has also zero right kernel according to the definition of standard filtrations. Thus it is nondegenerate and $\dim G'_1 = 2$.

LEMMA 4.5 ($p = 3$). *The graded Lie algebra $H_2''(\mathbf{1})$ has no nontrivial filtered deformations.*

Proof. Let L be a filtered deformation of $H_2''(\mathbf{1})$. As we have noted in the proof of Proposition 4.4, Theorem 2.1(1) applies to L. Let P be the subspace of L afforded by this theorem and $Q := N_{L_0}(P)$. Since $Q + L_1 = L_0$ and $Q \cap L_1 = L_2 = 0$, we have $Q \cong G_0 \cong \mathrm{sl}(2)$. The Q-module P has two composition factors $L_1 \cong G_1$ and $P/L_1 \cong G_{-1}$, both of which are two-dimensional. If e, h, f is a standard basis for Q, then $-1, 1$ are the only eigenvalues of h on P. Hence e^2 and f^2 annihilate P. By [7, Theorem 1] P is completely reducible. Let $V \subset P$ be a Q-submodule complementary to L_1. The multiplication in L gives rise to Q-module homomorphisms $\wedge^2 V \to L$ and $V \otimes L_1 \to L$. Since L contains no one-dimensional Q-submodules, we conclude $[V, V] = 0$. On the other hand, $V \otimes L_1 \cong V \otimes V \cong S^2(V) \oplus \wedge^2 V$ is a sum of a three-dimensional and a one-dimensional irreducible Q-modules. It follows $[V, L_1] \subset Q$. Thus the decomposition $V \oplus Q \oplus L_1$ is a gradation of L compatible with its filtration, i.e., L is a trivial deformation.

PROPOSITION 4.6 ($p = 3$). *Let $L = L'_{-2} \supset L'_{-1} \supset L'_0 \supset \dots$ be a standard separating filtration in a Lie algebra L. Suppose that the associated graded algebra G' satisfies conditions (1)–(3) of Proposition 4.4. If $L_0 \subset L$ is a subspace such that $L'_{-1} \supset L_0 \supset L'_0$ and $\dim L_0/L'_0 = 1$ then L_0 is a subalgebra and the graded algebra G associated with the standard filtration determined by the pair L, L_0 satisfies $X'(m_1) \subset G \subset X(m_1)$ for some m_1. Denote by $\mathfrak{h}'$ the normalizer of L'_0 in the universal p-envelope $P(L)$. If $[\mathfrak{h}', L_0] \subset L_0$ then there exists a hamiltonian form ω on $W_2(\mathbf{m})$ and an embedding of Lie algebras $\tau : L \to H_2(\mathbf{m}, \omega)$ such that $L_0 = \tau^{-1}(H_2(\mathbf{m}, \omega)_0)$ and $\tau(L) \supset H_2''(\mathbf{m}, \omega)$, where $\mathbf{m} = (m_1, 1)$. There exists at least one subalgebra L_0 satisfying the above properties. Thus L is hamiltonian.*

Proof. Since $\mathrm{gr}'_{-1} L_0$ is G'_0-invariant, L_0 is L'_0-invariant. Since in addition $\dim L_0/L'_0 = 1$, we have $[L_0, L_0] = [L'_0, L_0] \subset L_0$.

Put $\mathfrak{g}_{ir} = \mathrm{gr}_i(\mathrm{gr}'_r L) \cong \mathrm{gr}'_r(\mathrm{gr}_i L)$. The multiplication in L induces bilinear mappings $\mathfrak{g}_{ir} \times \mathfrak{g}_{js} \to \mathfrak{g}_{i+j, r+s}$. In particular, $\mathrm{gr}' G_{-1} = \mathfrak{g}_{-1,-2} \oplus \mathfrak{g}_{-1,-1}$ is a graded module over a graded Lie algebra $\mathrm{gr}' G_0 = \mathfrak{g}_{0,-1} \oplus \mathfrak{g}_{00} \oplus \mathfrak{g}_{01}$. As it follows from the faithfulness of G_0 on G_{-1}, the component $\mathfrak{g}_{01}$ acts faithfully on $\mathrm{gr}' G_{-1}$ and $\mathfrak{g}_{0r} = 0$ for all $r > 1$. Note that $\mathfrak{g}_{-1,-2} \cong L'_{-2}/L'_{-1}$, $\mathfrak{g}_{-1,-1} \cong L'_{-1}/L_0$, and $\mathfrak{g}_{0,-1} \cong L_0/L'_0$ are one-dimensional. In particular, $\dim G_{-1} = 2$. The properties of G' ensure that the mappings $\mathfrak{g}_{0,-1} \times \mathfrak{g}_{-1,-1} \to \mathfrak{g}_{-1,-2}$, and $\mathfrak{g}_{01} \times \mathfrak{g}_{-1,-2} \to \mathfrak{g}_{-1,-1}$ are nonzero, and that $\mathfrak{g}_{00}$ contains an element which acts as -1 on $\mathfrak{g}_{-1,-1}$ and as 1 on $\mathfrak{g}_{-1,-2}$. In particular, $\mathfrak{g}_{00} \cong G'_0$ is one-dimensional. Thus $\mathrm{gr}' G_0$ acts faithfully on $\mathrm{gr}' G_{-1}$, and in fact $\mathrm{gr}' G_0 \cong \mathrm{sl}(\mathrm{gr}' G_{-1})$. It follows $G_0 = [G_0, G_0]$, whence $G_0 \cong \mathrm{sl}(G_{-1})$. Since $\dim G = \dim G' > 5$, we get $G_1 \neq 0$. Then G is hamiltonian, $H_2''(\mathbf{m}) \subset G \subset H_2(\mathbf{m})$. Choose a subspace $M \subset L$ such that $L'_{-1} \supset M \supset L'_0$, $\dim M/L'_0 = 1$, $M \neq L_0$. As we have proved, M is a subalgebra of codimension 2 in L. Then $\mathrm{gr}\, M$ is a homogeneous subalgebra of codimension 2 in G. Since $M \cap L_0 = L'_0$, the component $\mathrm{gr}_{-1} M \cong M/L'_0$ is one-dimensional. Adjusting coordinates, we may assume $\mathrm{gr}_{-1} M = \mathscr{D}(kx_1)$. Then $\mathrm{gr}_0 M \subset N_{G_0}(\mathrm{gr}_{-1} M) = \mathscr{D}(kx_1x_2 + kx_1^{(2)})$, and $\mathrm{gr}_1 M \subset \mathscr{D}(kx_1x_2^{(2)} + kx_1^{(2)}x_2 + kx_1^{(3)})$ because $[\mathrm{gr}_{-1} M, \mathrm{gr}_1 M] \subset \mathrm{gr}_0 M$. We must have $\mathrm{gr}_1 M = G_1$, in particular $m_2 = 1$. Then $G \subset X(m_1)$. Note also that $\mathrm{gr}_0 L'_0 = \mathrm{gr}_0 M = \mathscr{D}(kx_1x_2 + kx_1^{(2)})$.

The remaining part of Proposition 4.6 follows from Lemma 4.5 when $G = H_2''(\mathbf{1})$. We therefore assume $G \neq H_2''(\mathbf{1})$. Denote by v' the height function on L associated with L'_0. We claim that $v'(D) \leq 1$ for all $D \in L'_{-1}$ and $v'(D) \leq m_1$ for all $D \in L$. Suppose first $D \in L_0$. Since the algebra $G_0 \cong \mathrm{sl}(2)$ has only inner derivation, $[D^3 - D', L_0] \subset L_1$ for a suitable $D' \in L_0$. As $L_1 \subset L'_0 \subset L_0$. we get $[D^3 - D', L'_0] \subset L'_0$, whence $v'(D) \leq 1$. This inequality holds actually for all $D \in L'_{-1}$ since we can always choose a subalgebra L_0 containg D. Let now $D \in L$ be arbitrary. We consider the elements $u \in P(L)$ such that $u \equiv D^{p^{m_1}} \pmod{P(L)_{m_1 - 1}}$. If $\overline{D}$ is the image of D in G_{-1}, then $(\mathrm{ad}\, \overline{D})^{p^{m_1}} = 0$. Hence $[u, L_i] \subset L_{i - p^{m_1} + 1}$ for all i and any such a u. We choose u with the property that the number t such that $[u, L_i] \subset L_{i+t}$ for all i is the maximal possible. In particular, $t > -p^{m_1}$. The inner derivation $\mathrm{ad}\, u$ induces a derivation $\Delta : G \to G$ of degree t. By Proposition 4.3, $\mathfrak{D}_l(G) = 0$ for $l < -1$. Hence, were $t < -1$, we could increase t by altering u. Thus, the maximality of t yields $t \geq -1$. If $t \geq 0$ then $[u, L_0] \subset L_0$ and $[u, L_1] \subset L_1$. Suppose $t = -1$. According to Proposition 4.3(2), we may assume that $\mathrm{gr}_0 L'_0 + G_+$ is stable under Δ, subtracting from u a suitable element of L_{-1}. This means $[u, L'_0] \subset L_0$ and $[u, L_1] \subset L'_0 + L_1 = L'_0$. Thus, in each case, $\mathrm{ad}\, u$ induces a mapping $L'_0/L_1 \to L_0/L'_0$ which is clearly a 1-cocycle. Now $L'_0/L_1 \cong \mathrm{gr}_0 L'_0$ is a two-dimensional subalgebra of G_0 and $L_0/L'_0 \cong G_0/\mathrm{gr}_0 L'_0$. Since all eigenvalues of $\mathscr{D}(x_1x_2)$ in G_0 are distinct, $H^1(\mathrm{gr}_0 L'_0, G_0/\mathrm{gr}_0 L'_0)$ vanishes. Thus, subtracting from u a suitable element of L_0, we achieve $[u, L'_0] \subset L'_0$, i.e., $u \in \mathfrak{h}'$. Then $v'(D) \leq m_1$ according to the definition of v'.

Suppose that $[\mathfrak{h}', L_0] \subset L_0$, i.e., $\mathfrak{h}' \subset N_{P(L)}(L_0)$. It is immediate that $v(D) \leq v'(D)$, where v is the height function on L associated with L_0. We see that $v(D) \leq m_1$ for $D \in L \setminus L'_{-1}$ and $v(D) \leq 1$ for $D \in L'_{-1} \setminus L_0$, i.e., $\mathbf{m}(L, L_0) \leq \mathbf{m}$. The opposite inequality is always true, thus $\mathbf{m}(L, L_0) = \mathbf{m}$. Theorem 3.4 shows that L is hamiltonian.

The multiplication in $P(L)$ induces an L'_0-invariant bilinear mapping $\mathfrak{h}'/L'_0 \times G'_{-1} \to G'_{-2}$. Since L'_0 acts trivially in $\mathfrak{h}'/L'_0$, it corresponds to a mapping $\mathfrak{h}'/L'_0 \to \mathrm{Hom}_{G'_0}(G'_{-1}, G'_{-2}) = 0$. In other words, $[\mathfrak{h}', L'_{-1}] \subset L'_{-1}$. Hence G'_{-1} is an $\mathfrak{h}'$-module. Denote by ρ' the representation of $\mathfrak{h}'$ in G'_{-1}. As $[\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}' \cap L = L'_0$, the commutant $[\rho'(\mathfrak{h}'), \rho'(\mathfrak{h}')]$ contains only scalar transformations. Thus $\rho'(\mathfrak{h}')$ is nilpotent. Since irreducible representations of a nilpotent Lie algebra have dimension a power of characteristic, ρ' is reducible. There exists a one-dimensional $\mathfrak{h}'$-invariant subspace of G'_{-1}, and we always can find an $\mathfrak{h}'$-invariant L_0.

Proof of Theorem 4.1. Under hypotheses of the theorem G is hamiltonian, $H''_2(\mathbf{m}) \subset G \subset H_2(\mathbf{m})$ where $\mathbf{m} = (m_1, m_2)$. If G does not occur as an exceptional case in Proposition 4.3 then $\mathscr{D}(G) = 0$. By Theorem 3.5, $\mathbf{m}(L, L_0) = \mathbf{m}(G, G_0 + G_+) = \mathbf{m}$. Then L is a filtered hamiltonian algebra by Theorem 3.4. Suppose $p = 3$, $m_2 = 1$. If $G = H''_2(\mathbf{1})$ we apply Lemma 4.5. Suppose (2) of Proposition 4.3 holds. Applying Propositions 4.4 and 4.6, we deduce that L is isomorphic to a hamiltonian Lie algebra $\tilde{L}$ as an abstract algebra. Suppose $G = H''_2(\mathbf{m})$, $\mathbf{m} \neq \mathbf{1}$. Then $\dim \tilde{L} = \dim L = \dim G = 3^m - 2$ where $m = m_1 + m_2$. Examining dimensions of hamiltonian algebras, we get $\omega = dx_1 \wedge dx_2$ and $\tilde{L} = H''_2(\mathbf{m})$. As is shown in [18, II, Sect. 7] (we return to this question at the end of present paper), all subalgebras of codimension 2 in $H''_2(\mathbf{m})$ are conjugate to each other. Thus, composing an isomorphism $\tau : L \xrightarrow{\sim} \tilde{L}$ with a suitable automorphism of $\tilde{L}$, we may assume that τ satisfies $\tau(L_0) = \tilde{L}_0$. Then τ is an isomorphism of filtered algebras.

5. TORAL RANK OF A FILTERED LIE ALGEBRA

For a restricted Lie algebra L denote by $\mathrm{MT}(L)$ the maximum of dimensions of tori contained in L. If L is an arbitrary Lie algebra then its toral rank $\mathrm{TR}(L)$ is $\mathrm{MT}(\mathrm{Der}' L)$ where $\mathrm{Der}' L$ is the restricted subalgebra of the derivation algebra $\mathrm{Der}\, L$ generated by $\mathrm{ad}\, L$. This definition agrees with the one given by H. Strade [22, 23]. A filtered restricted Lie algebra is by definition a restricted Lie algebra L endowed with a filtration $(L_i)_{i \in \mathbb{Z}}$ such that $u^p \in L_{pi}$ for every $i \in \mathbb{Z}$ and $u \in L_i$. The graded algebra associated with a filtered restricted Lie algebra carries the induced p-structure [26, Theorem 3.1].

THEOREM 5.1. *Let L be a filtered Lie algebra and G its associated graded algebra. Then* $\mathrm{TR}(L) \geq \mathrm{TR}(G)$.

Proof. Consider the filtration of $\mathrm{Der}\, L$ defined by the rule $\mathrm{Der}_i\, L := \{D \in \mathrm{Der}\, L | D(L_j) \subset L_{i+j} \text{ for all } j\}$, $i \in \mathbb{Z}$. Endowed with this filtration, $\mathrm{Der}\, L$ is a filtered restricted Lie algebra. There is a natural injective homomorphism of graded restricted Lie algebras $\nu : \mathrm{gr}\, \mathrm{Der}\, L \to \mathrm{Der}\, G$. If $\bar{u} \in G_i$ and $u \in L_i$ is a representative of $\bar{u}$ then $\nu((\mathrm{ad}\, u)^{p^r} + \mathrm{Der}_{p^r i+1} L)$

$= (\operatorname{ad} \bar{u})^{p^r}$. It follows that $\nu(\operatorname{gr} \operatorname{Der}' L) \supset \operatorname{Der}' G$. Hence $\mathrm{MT}(\operatorname{gr} \operatorname{Der}' L) \geq \mathrm{MT}(\operatorname{Der}' G) = \mathrm{TR}(G)$. The proposition below gives also the inequality $\mathrm{TR}(L) = \mathrm{MT}(\operatorname{Der}' L) \geq \mathrm{MT}(\operatorname{gr} \operatorname{Der}' L)$.

PROPOSITION 5.2. *Let L be a filtered restricted Lie algebra and G its associated graded algebra. Then* $\mathrm{MT}(L) \geq \mathrm{MT}(G)$.

The main idea of the proof is that MT behaves like a semicontinuous function, when applied to restricted Lie algebras in a parametric family. To realize this idea we need to introduce certain characteristic polynomials. Suppose that K is a commutative associative unital k-algebra and $\mathscr{L}$ a restricted Lie K-algebra which is free of finite rank as a K-module. By the Poincaré–Birkhoff–Witt theorem the restricted universal enveloping algebra $u(\mathscr{L})$ is also free of finite rank over K. Let ρ denote the left regular representation of $\mathscr{L}$ in $u(\mathscr{L})$. For each $u \in \mathscr{L}$ there is a well-defined characteristic polynomial of $\rho(u)$,

$$\chi_{\mathscr{L}}(u;\tau) := \det(\tau \cdot \mathrm{Id} - \rho(u)) = \sum_{i=0}^{p^n} f_i(u)\tau^i \in K[\tau],$$

where τ is an indeterminate, $n = \operatorname{rk}_K \mathscr{L}$. Obviously, the f_i's are homogeneous polynomial functions on $\mathscr{L}$. If $K \to K'$ is a homomorphism of algebras then $u(K' \otimes_K \mathscr{L}) \cong K' \otimes_K u(\mathscr{L})$ and the left regular representation of $K' \otimes_K \mathscr{L}$ in $u(K' \otimes_K \mathscr{L})$ is the extension of ρ. It follows that the coefficient functions of $\chi_{K' \otimes_K \mathscr{L}}$ are extensions of the functions f_i.

LEMMA 5.3. *Let L be a restricted Lie algebra over k. Then χ_L is a p-polynomial, i.e., $f_i = 0$ unless i is a power of p. Furthermore,* $\mathrm{MT}(L)$ *equals the largest number r such that $f_{p^{n-r}} \neq 0$ where $n = \dim L$.*

Proof. Let $u \in L$, and let M be the restricted subalgebra of L generated by u. If $m = \dim M$ then $u(L)$ is a free $u(M)$-module of rank p^{n-m}. It follows $\chi_L(u;\tau) = \chi_M(u;\tau)^{p^{n-m}}$. The elements u^{p^j}, $0 \leq j < m$, form a basis for M and $\sum_{j=0}^m a_j u^{p^j} = 0$ for some $a_j \in k$, $a_m = 1$. Computing in the basis u^i, $0 \leq i < p^m$, of $u(M)$, we get at once $\chi_M(u;\tau) = \sum_{j=0}^m a_j \tau^{p^j}$. Hence, $f_i(u)$ is a power of a_j if $i = p^{n-m+j}$, $0 \leq j \leq m$, and $f_i(u) = 0$ otherwise. This proves the first assertion.

Suppose that $T \subset L$ is a torus. Then T is generated as a restricted Lie algebra by a single element $u \in T$. In the notations of the preceding paragraph we have $M = T$. Furthermore, $a_0 \neq 0$ since u is semisimple. Hence $f_i(u) \neq 0$ for $i = p^{n-m}$. It follows $\dim T = m \leq r$.

Conversely, choose $u \in L$ such that $f_i(u) \neq 0$ for $i = p^{n-r}$. Then $m \geq r$ and $a_{m-r} \neq 0$. Moreover, $a_j = 0$ for $j < m - r$ since $f_i(u) = 0$ for $i < p^{n-r}$. It follows that $u^{p^{m-r}}$ is semisimple and the torus T generated by this element has u^{p^j}, $m - r \leq j < m$, as its basis. Thus $\dim T = r$.

Remark. The characterization of $\mathrm{MT}(L)$ in terms of p-polynomials is due to A. Premet [14, Theorem 2]. However, the paper of Premet deals mainly with minimal p-polynomials while it is necessary for our purposes to work with characteristic ones.

A continuous family of restricted Lie algebras parametrized by an affine algebraic variety X is, by definition, a restricted Lie algebra $\mathscr{L}$ over the algebra $k[X]$ of regular functions on X such that $\mathscr{L}$ is free (more generally, projective) of finite rank over $k[X]$. The individual member of the family corresponding to a point $x \in X$ is the restricted Lie algebra $\mathscr{L}/\mathfrak{m}_x\mathscr{L}$ where $\mathfrak{m}_x$ is the maximal ideal of $k[X]$ consisting of all functions vanishing at x. Let L, G be two restricted Lie algebras. We call G a contraction of L if there exists a continuous family $\mathscr{L}$ of restricted Lie algebras parametrized by an irreducible affine algebraic variety X such that $\mathscr{L}/\mathfrak{m}_x\mathscr{L} \cong L$ for all x in a nonempty open subset of X and $\mathscr{L}/\mathfrak{m}_y\mathscr{L} \cong G$ for at least one point $y \in X$, all isomorphisms being understood as isomorphisms of restricted Lie algebras.

PROPOSITION 5.4. *If G is a contraction of L then* $\mathrm{MT}(G) \le \mathrm{MT}(L)$.

Proof. Let $X, \mathscr{L}$ be those defining the given contraction. Consider the characteristic polynomial $\chi_{\mathscr{L}} = \sum_i f_i \tau^i$. For each $x \in X$ let $\alpha_x : k[X] \to k$ denote the homomorphism with kernel $\mathfrak{m}_x$ and $\pi_x : \mathscr{L} \to \mathscr{L}/\mathfrak{m}_x\mathscr{L}$ the canonical projection. According to the functoriality of characteristic polynomials we have $\chi_{\mathscr{L}/\mathfrak{m}_x\mathscr{L}}(\pi_x(u); \tau) = \sum_i \alpha_x(f_i(u))\tau^i$ for all $u \in \mathscr{L}$. Now Lemma 5.3 yields

$$\{x \in X | \mathrm{MT}(\mathscr{L}/\mathfrak{m}_x\mathscr{L}) \ge r\} = \{x \in X | f_i(\mathscr{L}) \not\subset \mathfrak{m}_x \text{ for some } i \le p^{n-r}\},$$

where $n = \mathrm{rk}_{k[X]}\mathscr{L}$. It follows that this is an open subset of X for any r. Take $r = \mathrm{MT}(G)$. Then the subset is nonempty, hence it contains a point x such that $\mathscr{L}/\mathfrak{m}_x\mathscr{L} \cong L$. It follows $\mathrm{MT}(L) \ge r$.

LEMMA 5.5. *Let L be a filtered restricted Lie algebra, $G = \operatorname{gr} L$. Suppose that the filtration in L is exhaustive and separating. Then there exists a restricted Lie algebra $\mathscr{L}$ over the polynomial algebra $k[t]$ such that $\mathscr{L}$ is free of finite rank over $k[t]$ and there are isomorphisms of restricted Lie algebras $\mathscr{L}/t\mathscr{L} \cong G$ and $\mathscr{L}/(t-\lambda)\mathscr{L} \cong L$ for every nonzero $\lambda \in k$.*

Proof. For each i choose a subspace $V_i \subset L_i$ complementary to L_{i+1}. Consider the restricted Lie algebra $L[t, t^{-1}] := k[t, t^{-1}] \otimes L$ over the ring of Laurent polynomials $k[t, t^{-1}]$ obtained from L by extension of scalars. Identify L with a k-subalgebra of $L[t, t^{-1}]$. Define an invertible $k[t, t^{-1}]$-linear transformation θ of $L[t, t^{-1}]$ by the rule $\theta(u) = t^i u$ for $u \in V_i$. Then θ is an isomorphism of $L[t, t^{-1}]$ onto a new restricted Lie $k[t, t^{-1}]$-

algebra $L[t,t^{-1}]_\theta$ having the same underlying $k[t,t^{-1}]$-module, a new multiplication $[u,v]_\theta := \theta[\theta^{-1}u, \theta^{-1}v]$, and a new pth power mapping $u^{[p]_\theta} := \theta((\theta^{-1}u)^p)$, $u, v \in L[t,t^{-1}]$. If $u \in V_i$, $v \in V_j$ then

$$[u,v]_\theta \equiv \mu(u,v), \qquad u^{[p]_\theta} \equiv \delta(u) \pmod{tk[t]L},$$

where $\mu(u,v)$ is the component of $[u,v]$ lying in V_{i+j} and $\delta(u)$ is the component of u^p lying in V_{pi}. In particular, $[u,v]_\theta$ and $u^{[p]_\theta}$ belong to $k[t]L$. In other words, $\mathscr{L} := k[t]L$ is a restricted $k[t]$-subalgebra of $L[t,t^{-1}]_\theta$. It is immediate from the congruence relations above that $\mathscr{L}/t\mathscr{L} \cong G$. If $0 \neq \lambda \in k$ then the homomorphism $k[t] \to k$ sending t to λ extends to a homomorphism $k[t,t^{-1}] \to k$. It follows

$$\begin{aligned}\mathscr{L}/(t-\lambda)\mathscr{L} &\cong L[t,t^{-1}]_\theta/(t-\lambda)L[t,t^{-1}]_\theta \\ &\cong L[t,t^{-1}]/(t-\lambda)L[t,t^{-1}] \cong L.\end{aligned}$$

Proof of Proposition 5.2. Since $\overline{L} := \cup L_i / \cap L_i$ has the same associated graded algebra as L and $\mathrm{MT}(L) \geq \mathrm{MT}(\overline{L})$, we may assume without loss of generality that the filtration of L is exhaustive and separating. Then Lemma 5.5. shows that G is a contraction of L, and Proposition 5.4 applies.

6. DETERMINATION OF TORAL RANK ONE ALGEBRAS

LEMMA 6.1. *Let L be a Lie algebra, $T \subset \operatorname{Der} L$ a one-dimensional torus, $H := L^T \subset L$ the subalgebra of elements annihilated by T, and $M \subset L$ a maximal T-invariant subalgebra containing H. If $p = 2$ then L/M is an irreducible H-module. If $p = 3$ then L/M is either an irreducible H-module or a sum of two T-invariant irreducible H-modules which correspond to two opposite weights with respect to T.*

Proof. If $p = 2$ then we have weight space decompositions $L = H \oplus L_\alpha$ and $M = H \oplus M_\alpha$ with respect to T. Since M is a maximal T-invariant subalgebra, M_α is a maximal H-submodule of L_α. Hence $L/M \cong L_\alpha/M_\alpha$ is H-irreducible.

If $p = 3$ then $L = L_{-\alpha} \oplus H \oplus L_\alpha$ and $M = M_{-\alpha} \oplus H \oplus M_\alpha$. Suppose that $M_\alpha \neq L_\alpha$. Let $N_\alpha \subset L_\alpha$ be a maximal H-submodule containing M_α. Put $N_{-\alpha} := M_{-\alpha} + [N_\alpha, N_\alpha]$. Then $[N_{-\alpha}, N_{-\alpha}] \subset [M_{-\alpha}, M_{-\alpha}] + [L_{-\alpha}, [N_\alpha, N_\alpha]] \subset M_\alpha + [H, N_\alpha] \subset N_\alpha$. Hence $N := N_{-\alpha} \oplus H \oplus N_\alpha$ is a proper T-invariant subalgebra of L containing M. It follows $M = N$, i.e., M_α is a maximal H-submodule of L_α.

By symmetry $M_{-\alpha} = L_{-\alpha}$ or $M_{-\alpha}$ is a maximal H-submodule of $L_{-\alpha}$. We see that $L/M \cong L_{-\alpha}/M_{-\alpha} \oplus L_\alpha/M_\alpha$ is a sum of at most two irreducible H-modules.

Suppose that $(L_i)_{i\in\mathbb{Z}}$ is a standard filtration in a Lie algebra L and L_0 contains a Cartan subalgebra H of L. Since the subspaces of filtration are H-invariant, H operates in the homogeneous components $G_i := L_i/L_{i+1}$ of the associated graded algebra $G := \operatorname{gr} L$. We have weight space decompositions $G_i = \bigoplus_\alpha G_{i,\alpha}$ with respect to H, where α runs through the set of roots. If L_β is the root space of L corresponding to a root β, then $G_{i,\beta} \cong L_i \cap L_\beta/L_{i+1} \cap L_\beta$. Note that $\operatorname{gr} L_\beta = \bigoplus_i G_{i,\beta}$. If G_{-1} is an irreducible H-submodule then $G_{-1} = G_{-1,-\alpha}$ for a certain root $-\alpha$. According to (g1) and (g2), G_i is a quotient of $G_{i+1} \otimes G_{-1}$ for $i < -1$, while G_i is a submodule of $\operatorname{Hom}(G_{-1}, G_{i-1})$ for $i > -1$. Proceeding by induction, we conclude $G_i = G_{i,i\alpha}$ for all i. In particular, G_0 is a factor algebra of H in this case. To avoid confusion we shall denote the zero root with respect to H as $\bar{0}$.

LEMMA 6.2. *If a Lie algebra L contains a nilpotent maximal subalgebra H then $[L, L] \neq L$.*

Proof. If the normalizer $N_L(H)$ is distinct from H, then there exists a subalgebra of L containing H as an ideal of codimension 1. By the maximality of H this subalgebra coincides with L. It follows $[L, L] \subset H$. We thus may assume that H is selfnormalizing in L, i.e., a Cartan subalgebra.

Suppose that $-\alpha$ is a nonzero root with respect to H and $V \subset L_{-\alpha}$ an irreducible H-module. Consider the graded algebra G associated with the standard filtration of L determined by $L_0 := H$ and $L_{-1} := V + H$. Here $G_{-1} \cong V$ is an irreducible H-module with eigenvalue function $-\alpha$. Hence $G_i = G_{i,i\alpha}$ for all i. It follows $\operatorname{gr} L_\beta = \bigoplus_{i\alpha=\beta} G_i$ for each root β. In particular, $\bigoplus_{i\geq 0} G_i = \operatorname{gr} H = \bigoplus_{i\equiv 0 (\operatorname{mod} p)} G_i$. We deduce $G_{-p} = 0$ and $G_1 = 0$, whence $G_i = 0$ whenever $i \leq -p$ or $i \geq 1$. Since L_0 is a maximal subalgebra, the filtration of L is exhaustive. If β is a nonzero root then $L_\beta \cap L_0 = 0$, whence the filtration induced on L_β is separating. It follows then that $\beta = i\alpha$ for some i, $1 - p \leq i < 0$, and $\operatorname{gr} L_\beta = G_i$, i.e., $L_\beta \cong G_i$. In particular, $L_{-\alpha} \cong G_{-1}$ is H-irreducible. In view of (g2), $L_{i\alpha} = [L_{-\alpha}, L_{(i+1)\alpha}]$ for all $1 - p \leq i \leq -2$. By induction on i we get $[L_{i\alpha}, L_{-i\alpha}] \subset [L_\alpha, L_{-\alpha}]$ for all $1 - p \leq i \leq -2$. As the root $-\alpha$ in the preceding discussion was an arbitrary one, L_β is H-irreducible for every nonzero root β.

Suppose $[L, L] = L$. Then $H = [H, H] + [L_\alpha, L_{-\alpha}]$. Since H is nilpotent, it is generated by any subspace complementary to $[H, H]$. As $[L_\alpha, L_{-\alpha}]$ is an ideal of H, it follows $H = [L_\alpha, L_{-\alpha}]$. In particular

$L_\alpha \neq 0$, i.e., α is a root. Then L_α is H-irreducible. By Lemma 1.6, $\dim H/[H, H] = 1$, whence $\dim H = 1$. The irreducibility of root spaces yields $\dim L_\alpha = 1$. Hence $H + L_\alpha$ is a solvable subalgebra. Then L must coincide with it, and so be solvable, a contradiction.

THEOREM 6.3. *Let L be a simple Lie algebra containing a Cartan subalgebra H of toral rank 1 in L. Let L_0 be a maximal subalgebra of L containing H and Γ the representation of L_0 in L/L_0. If $p = 3$ then one of the following two possibilities holds*:

(1) $\dim L/L_0 = 1$, $L \cong W_1(m)$ *for some* m;

(2) $\dim L/L_0 = 2$, $\Gamma(L_0) = \mathrm{sl}(L/L_0)$, $L \cong H_2''(\mathbf{m}, \omega)$ *for some* $\mathbf{m}$.

If $p = 2$ then one of the following three possibilities holds:

(3) $\dim L/L_0 = 1$, $L \cong K_1'(m) := [W_1(m), W_1(m)]$ *for some* $m > 1$;

(4) $\dim L/L_0 = 2$, $\Gamma(L_0) = \mathrm{sl}(L/L_0)$, $L \cong H_2''(\mathbf{m}, \omega)$ *for some* $\mathbf{m}$;

(5) $\dim L/L_0 = 4$. *In this case the graded algebra G associated with the standard filtration of L determined by L_0 and $L_{-1} := L$ is semisimple and contains a single minimal ideal $B \otimes S$ where $B = O_1(2)$, $S = K_1'(m)$, $m > 1$, or $B = O_1(1)$, $S = H_2''(\mathbf{m})$, $m_1, m_2 > 1$. Furthermore, $\pi(G) = k\partial$ where $\pi : G \to \operatorname{Der} B$ is the canonical homomorphism and ∂ is the special derivation of B sending $x^{(r)}$ to $x^{(r-1)}$, $r > 0$.*

Remark. If $p = 2$ and $\mathbf{m} = (m_1, 1)$ then $H_2''(\mathbf{m}, \omega)$ is not necessarily simple.

Proof. Choose an L_0-submodule $L_{-1} \supset L_0$ such that L_{-1}/L_0 is an irreducible L_0-module and consider the corresponding standard filtration. The associated graded algebra G satisfies (g1)–(g3). Note that $H = L^T$ where T is the maximal torus of the restricted subalgebra of $\operatorname{Der} L$ generated by $\mathrm{ad}_L H$. The root spaces relative to H coincide with the weight spaces relative to T. By the definition of toral rank, $\dim T = 1$. Hence Lemma 6.1 applies. We have three possibilities:

(A) L/L_0 is H-irreducible;

(B) $p = 3$, L/L_0 is a direct sum of two irreducible H-modules, L/L_0 is L_0-reducible;

(C) $p = 3$, L/L_0 is a direct sum of two irreducible H-modules, L/L_0 is L_0-irreducible.

Consider first cases (A) and (B). In case (A), $G_{-2} = 0$ and G_{-1} is an irreducible H-module. In case (B), $L_{-1} \neq L$. Then G_{-1} must be isomorphic to one of the two summands in the decomposition of L/L_0 and G_{-2} to the other. Both G_{-1} and G_{-2} are H-irreducible, $G_{-3} = 0$. By the remarks preceding Lemma 6.2, G_0 is a homomorphic image of H, hence nilpotent, in both cases. Were $G_1 = 0$, then $L_0 \cong G_0$ would be nilpotent, which contradicts Lemma 6.2. Hence $G_1 \neq 0$. We claim that G satisfies (g4) as well. Consider $C_G(G_+) \cap G_-$. It is a homogeneous G_0-submodule. If it is nonzero it must contain G_{-1}, or in case (B), G_{-2}. However, $[G_{-1}, G_1] \neq 0$ according to (g1), (g2). In case (B), G_{-2} is a faithful G_0-module, whence $[G_{-2}, [G_{-1}, G_1]] \neq 0$. It follows $[G_{-2}, G_1] \neq 0$ as well. Thus $C_G(G_+) \cap G_-$ is zero. Now Theorem 1.2 applies, $A(G) = B \otimes S$. By Lemma 1.5, S satisfies the hypotheses of Theorem 1.1. In particular S, hence also G, has depth 1, i.e., case (B) is impossible.

Suppose first $B = k$. Then $G_{-1} = S_{-1}$. If $\dim G_{-1} = 1$ then either $G \cong W_1(m)$ or $p = 2$, $G \cong K_1'(m)$. These algebras have no nontrivial filtered deformations [4; 13, Corollary 3.3]. As L is simple, we conclude $L \cong W_1(m)$ for $p = 3$, or $L \cong K_1'(m)$, $m > 1$, for $p = 2$. Otherwise $p = 2$, $\dim G_{-1} = 2$, $S_0 \cong \mathrm{sl}(G_{-1}) \subset G_0$. Since G_0 is nilpotent, $G_0 \cong \mathrm{sl}(G_{-1})$. By Theorem 4.1, L is hamiltonian.

Suppose now $B \neq k$. We may view B as a G_0-invariant G_0-simple commutative subalgebra of the endomorphism algebra $\mathrm{End}\, G_{-1}$. Regarded as a B-module, G_{-1} is free of rank equal to $\dim S_{-1}$, i.e., 1 or 2. By Lemma 2.5, G_0 acts nilpotently on B. Let ρ be the representation of G_0 in G_{-1}. Then $G_0 \neq G_0'' := \rho^{-1}(\mathrm{End}_B G_{-1})$. Since G_0 is nilpotent, $G_0' := [G_0, G_0] + G_0'' \neq G_0$ as well. Note that $\rho(A(G)_0) \subset \mathrm{End}_B G_{-1}$. Hence $[G_{-1}, G_1] = A(G)_0 \subset G_0'$. A nilpotent algebra G_0 has nonzero center. Hence Lemma 2.3 and Theorem 2.1 apply. If $p = 3$ it follows $\mathrm{gr}_0[L, L] \subset G_0'$. However, $[L, L] = L$, and so $\mathrm{gr}_0[L, L] = G_0$, a contradiction. If $p = 2$, we get $H^0(G_0, H^{2,-1}(G)) = 0$, $H^1(G_0, K^1) = 0$, and $\mathrm{gr}_0[L, L_0] \subset G_0'$. By Lemma 2.2, $\dim G_0/G_0' = 1$ and there exists a nondegenerate skewsymmetric bilinear form on G_{-1}. Since G_0/G_0'' is nilpotent and has dimension 1 modulo its commutant, $\dim G_0/G_0'' = 1$ as well, i.e., $G_0' = G_0''$. Let $\pi : G \to \mathrm{Der}\, B$ be the canonical homomorphism. As $\ker \pi|_{G_0} = G_0'$, we have $\dim \pi(G_0) = 1$. By Lemma 2.4, $B \cong O_1(m)$ and $\pi(G_0) = k\partial$. Since components $G_i, i \neq 0$, annihilate B (see remarks following the statement of Theorem 2.2), $\pi(G) = \pi(G_0) = k\partial$. Apply now Proposition 2.6(2). If $Z^{20}(G) = Z^{20}(G')$ then $K^2 = 0$, whence $\mathrm{gr}_0[L, L] \subset G_0'$ by Theorem 2.1, a contradiction. We conclude that either $\dim S_{-1} = 1$, $m \leq 2$, or $\dim S_{-1} = 2$, $m = 1$. Under assumptions $\dim S_{-1} = 1$, $m = 1$ we have $\dim G_{-1} = 2$, $\dim G_0 = 3$. The existence of a nondegenerate bilinear form on G_{-1} ensures $G_0 \cong \mathrm{sl}(G_{-1})$. Hence G is hamiltonian.

We now turn to case (C). Here $G_{-2} = 0$. There are weight space decompositions $G_{-1} = G_{-1,-\alpha} \oplus G_{-1,\alpha}$ and $G_0 = G_{0,-\alpha} \oplus G_{0,\bar{0}} \oplus G_{0,\alpha}$. Here $G_{0,\bar{0}}$ is a homomorphic image of H, hence a toral rank one Cartan subalgebra of G_0. Put $J = J_{-\alpha} \oplus J_{\bar{0}} \oplus J_\alpha := G_{0,-\alpha} \oplus [G_{0,-\alpha}, G_{0,\alpha}] \oplus$

$G_{0,\alpha}$. Then $J \neq 0$ since otherwise $G_0 = G_{0,\bar{0}}$ wouldn't act irreducibly on G_{-1}. Components $G_{0,\pm\alpha}$ act nilpotently on G_{-1}. Suppose that I is an ideal of G_0. If $I_{\bar{0}} := I \cap G_{0,\bar{0}}$ acts nilpotently on G_{-1}, then so does the whole I by Engel's theorem. Since G_0 acts faithfully in G_{-1}, it will follow $I = 0$. Thus $\alpha(I_{\bar{0}}) \neq 0$ provided $I \neq 0$. In this case $G_{0,\pm\alpha} = [I_{\bar{0}}, G_{0,\pm\alpha}] \subset [I, J]$, whence $J \subset [I, J] \subset I$. Thus J is the smallest nonzero ideal of G_0. The preceding inclusion with $I = J$ shows also that $J = [J, J]$.

Since $[G_{0,\alpha}, G_{-1,-\alpha}] = 0$, it follows that $[G_{0,\alpha}, G_{0,\alpha}]$ annihilates G_{-1}. The faithfulness of G_0 on G_{-1} yields $[G_{0,\alpha}, G_{0,\alpha}] = 0$. We have similarly $[G_{0,-\alpha}, G_{0,-\alpha}] = 0$. Thus we may view G_0 as a $\mathbb{Z}$-graded algebra. Any proper $G_{0,\bar{0}}$-submodule $V \subset G_{0,\alpha}$ generates an ideal of G_0 properly contained in J. It follows $V = 0$, i.e., $G_{0,\alpha}$ is $G_{0,\bar{0}}$-irreducible. Similarly, so is $G_{0,-\alpha}$. Hence G_0 satisfies (g1)–(g4). By Theorem 1.2, $J = B \otimes S$ where B is the centroid of J and S a simple homogeneous subalgebra. In view of Lemma 1.5, S satisfies the hypotheses of Theorem 1.1 We conclude $S \cong W_1(m)$. As only components of degree $-1, 0, 1$ are present, $S \cong W_1(1) \cong \mathrm{sl}(2)$.

There is an isomorphism of $G_{0,\bar{0}}$-modules $G_{-1,\alpha} \cong J_\alpha$, both of them being irreducible with eigenvalue function α. Using this isomorphism, we define a B-module structure on $G_{-1,\alpha}$. Then $G_{-1,\alpha}$ is free of rank 1 over B and the multiplication mapping $J_{\bar{0}} \times G_{-1,\alpha} \to G_{-1,\alpha}$ is B-bilinear. Similarly, there is a B-module structure on $G_{-1,-\alpha}$ such that $G_{-1,-\alpha}$ is free of rank 1 over B and the multiplication mapping $J_{\bar{0}} \times G_{-1,-\alpha} \to G_{-1,-\alpha}$ is B-bilinear. For $x \in J_{-\alpha}, y \in J_\alpha, z \in G_{-1,\alpha}$ we have $[x,[y,z]] = [[x,y],z]$. Let $e_0 \in S_0$ be the element such that $\alpha(e_0) = 1$. If $f \in B$ then Jacobi identity applied to fe_0, y, z gives $f[y,z] = -[fy,z] - [y,fz]$. Applying now ad x, we get

$$[x, f[y,z]] = -[[x,fy],z] - [[x,y],fz] = f[[x,y],z] = f[x,[y,z]].$$

Thus $[x, ft] = f[x,t]$ for all $t \in [G_{0,\alpha}, G_{-1,\alpha}]$. Note that $[G_{0,\alpha}, G_{-1,\alpha}]$ is a nonzero $G_{0,\bar{0}}$-submodule of $G_{-1,-\alpha}$, hence it is the whole of $G_{-1,-\alpha}$. We deduce that $\rho(J_{-\alpha}) \subset \mathrm{End}_B G_{-1}$ where ρ is the representation of G_0 on G_{-1}. Similarly $\rho(J_\alpha) \subset \mathrm{End}_B G_{-1}$. Thus $\rho(J) \subset \mathrm{End}_B G_{-1}$. Since $[J,J] = J$, we have $\rho(J) \subset \mathrm{sl}_B(G_{-1})$. Both J and $\mathrm{sl}_B(G_{-1})$ are free of rank 3 over B (note that $\mathrm{rk}_B G_{-1} = 2$), hence they have equal dimensions. It follows $\rho(J) = \mathrm{sl}_B(G_{-1})$. Next, $\mathrm{End}_B G_{-1} = B \oplus \mathrm{sl}_B(G_{-1})$. Since $\rho^{-1}(B)$ is an ideal of G_0 having zero intersection with J, it is zero. Thus $J = \rho^{-1}(\mathrm{End}_B G_{-1})$. Put $G_0' := J + [G_0, G_0]$.

Let $\pi : G_0 \to \mathrm{Der}\, B$ be the canonical homomorphism. Then $J = \ker \pi$. Since B is isomorphic to a subalgebra of $\mathrm{End}\, G_{-1,\alpha}$, the action of $G_{0,\bar{0}}$ on B is nilpotent by Lemma 2.5. Hence so is the action of $G_0 = G_{0,\bar{0}} + J$ as well. We may apply now Propositions 2.6(1), 2.7. Thus $H^0(G_0, H^{2,-1}(G))$

$= 0$ and $Z^{10}(G) = Z^{10}(G_{-1} \oplus J)$, hence also $Z^{10}(G) = Z^{10}(G')$ where we put $G' := G_{-1} \oplus G'_0$. Since the elements of G_1 correspond to coboundaries in $Z^{10}(G)$, it follows in particular $[G_{-1}, G_1] \subset G'_0$. Now by Theorem 2.1(2), $\mathrm{gr}_0[L, L_0] \subset G'_0$. By Lemma 2.2, $\dim G_0/G'_0 \leq 1$. Since $G_0/J \cong G_{0,\bar{0}}/J_{\bar{0}}$ is nilpotent, $\dim G_0/J \leq 1$ as well, i.e., $G'_0 = J$. By Proposition 2.6(2), $Z^{20}(G) = Z^{20}(G')$, and by Theorem 2.1(3), $\mathrm{gr}_0[L, L] \subset G'_0$. Since $[L, L] = L$, we conclude $G_0 = J$. This means that G_0 is simple, $G_0 = S \cong \mathrm{sl}(2)$, and $B = k$. Furthermore, G_{-1} is a two-dimensional irreducible G_0-module. If $G_1 = 0$ then $L_0 \cong G_0 \cong \mathrm{sl}(2)$. Since the G_0-modules G_{-1}, G_0 correspond to different central characters, every extension of G_{-1} by G_0 splits. Thus there exists an L_0-submodule $V \subset L$ such that $L = V \oplus L_0$. Now $\bigwedge^2 V$ is a one-dimensional L_0-module, which does not occur in L. Hence $[V, V] = 0$. Then V is an ideal of L, a contradiction. Thus $G_1 \neq 0$. By Theorem 4.1, L is hamiltonian.

LEMMA 6.4. *If* $p = 3$ *then* $H''_2(\mathbf{1}) \cong \mathrm{psl}(3)$.

Proof. If Φ is the classical root system of type A_2 then $G := \mathrm{psl}(3)$ has decomposition $H \oplus \sum_{\alpha \in \Phi} L_\alpha$, where H is a one-dimensional Cartan subalgebra of G and L_α one-dimensional subspaces. Let α_1, α_2 be a basis of Φ. Put $G_{-1} := L_{-\alpha_2} + L_{-\alpha_1-\alpha_2}$, $G_0 := L_{-\alpha_1} + H + L_{\alpha_1}$, $G_1 := L_{\alpha_2} + L_{\alpha_1+\alpha_2}$. Then $G = G_{-1} \oplus G_0 \oplus G_1$ is a $\mathbb{Z}$-gradation. Here $\dim G_{-1} = \dim G_1 = 2$, $G_0 \cong \mathrm{sl}(G_{-1})$. Hence $G \cong H''_2(\mathbf{1})$.

THEOREM 6.5. *Any simple Lie algebra of absolute toral rank* 1 *and characteristic* 3 *is isomorphic to either* sl(2) *or* psl(3), *hence is classical. Any Lie algebra of absolute toral rank* 1 *and characteristic* 2 *is solvable.*

Proof. If L is a simple Lie algebra of absolute toral rank 1 then every one of its Cartan subalgebras has toral rank 1 in L, so Theorem 6.3 applies. Block and Wilson [3, 2.2.3] determined the absolute toral rank of those Cartan type Lie algebras that contain a Cartan subalgebra of toral rank 1. Note that the algebras $W_1(m)$, $K'_1(m)$ have toral rank 1 only when $m = 1$ (in case $p = 2$ the algebras $W_1(m)$ and $K'_1(m)$ have the same p-envelope). Only the graded algebra $H''_2(\mathbf{1})$ has toral rank 1 among the algebras $H''_2(\mathbf{m}, \omega)$. Thus if $p = 3$, then either $L \cong W_1(1) \cong \mathrm{sl}(2)$ or $L \cong H''_2(\mathbf{1}) \cong \mathrm{psl}(3)$.

Let $p = 2$. If there exists a nonsolvable Lie algebra of absolute toral rank 1, then there exists a simple Lie algebra L with this property. Consider a standard filtration of L and the associated graded algebra G as in the proof of Theorem 6.3. By Theorem 5.1, $\mathrm{TR}(G) \leq 1$. Now G contains a simple homogeneous subalgebra S isomorphic to either $K'_1(m), m > 1$, or $H''_2(\mathbf{m})$, $m_1, m_2 > 1$. Then $\mathrm{TR}(S) > 1$. On the other hand, $\mathrm{TR}(S) \leq \mathrm{TR}(G)$, a contradiction.

COROLLARY 6.6. *Every simple Lie algebra* L *of characteristic* 2 *and absolute toral rank* 2 *contains a solvable maximal subalgebra* L_0 *such that* L/L_0 *is an irreducible* L_0*-module.*

Proof. Consider root space decomposition of L with respect to a two-dimensional torus T in a p-envelope of L. If α is a nonzero root with respect to T, then toral rank one section $L(\alpha) := C_L(T) + L_\alpha$ has toral rank ≤ 1 [22, Theorem 2.6]. By Theorem 6.5 it is solvable. Note that $L(\alpha) = C_L(T_\alpha)$ where $T_\alpha := \{t \in T \mid \alpha(t) = 0\}$ is a one-dimensional subtorus of T. Let L_0 be a maximal subalgebra of L containing $L(\alpha)$ and G the graded algebra associated with the standard filtration of L determined by L_0, L. By Lemma 6.1, $G_{-1} := L/L_0$ is an irreducible $L(\alpha)$-module. The representation of T_α in G_{-1} has a unique nonzero weight. Since $G_0 \subset \operatorname{End} G_{-1}$, it follows that T_α annihilates G_0, i.e., G_0 is a homomorphic image of $L(\alpha)$, hence solvable. Since L_1 is nilpotent, L_0 is solvable as well.

EXAMPLE. Let $p = 2$. We shall construct an infinite family of simple Lie algebras satisfying (5) of Theorem 6.3. First we shall describe a certain infinite dimensional Lie algebra. Put $\overline{G} = k\partial_1 + O_2(2,\infty)\partial_2$. This is a subalgebra of the infinite dimensional Lie algebra $W_2(2,\infty)$. Consider the gradation of $\overline{G}$ induced by the type $(0,1)$ gradation of $W_2(2,\infty)$. Recall that the latter is determined by the assignments $\deg x_1 = 0$, $\deg x_2 = 1$. Identify $O_1(2)$ with the subalgebra of $O_2(2,\infty)$ spanned by $1, x_1, x_1^{(2)}, x_1^{(3)}$. Then the lowest homogeneous components of $\overline{G}$ are $\overline{G}_{-1} = O_1(2)\partial_2$ and $\overline{G}_0 = k\partial_1 + O_1(2)x_2\partial_2$. Note that $\overline{G}_{-1}$ is $\overline{G}_0$-irreducible. Furthermore, $\overline{G}_0$ is a nilpotent restricted subalgebra of linear transformations of $\overline{G}_{-1}$ with a one-dimensional maximal torus $x_2\partial_2$. Define linear transformations $\int_2, \varepsilon_2$ of $O_2(2,\infty)$ by the formulas

$$\int_2\left(x_1^{(r)}x_2^{(s)}\right) = x_1^{(r)}x_2^{(s+1)},$$

$$\varepsilon_2\left(x_1^{(r)}x_2^{(s)}\right) = 0 \quad \text{for } s > 0, \quad \varepsilon_2\left(x_1^{(r)}\right) = x_1^{(r)}.$$

Thus ε_2 has image $O_1(2)$. The composite mapping $\partial_1^3\varepsilon_2$ has image k. Define a skewsymmetric bilinear mapping $\varphi : \overline{G} \times \overline{G} \to \overline{G}$ as

$$\varphi(\partial_1, D) = 0 \qquad \text{for all } D \in \overline{G},$$

$$\varphi(f\partial_2, g\partial_2) = \partial_1^3\varepsilon_2(fg)\partial_1 + \left(\partial_1(f)\partial_1^3\int_2(g) + \partial_1(g)\partial_1^3\int_2(f)\right)\partial_2,$$

$f, g \in O_2(2,\infty)$. Using the ∂_1-invariance of φ and the identities $\partial_2 \circ \int_2 = \mathrm{id}$, $[\partial_2, \int_2] = \varepsilon_2$, one verifies straightforwardly that φ is a 2-cocycle. Using identities $\partial_1 \varepsilon_2 = \varepsilon_2 \partial_1$, $\partial_1 \int_2 = \int_2 \partial_1$, $\partial_1^4 = 0$, one verifies that φ satisfies the Jacobi identity. Hence the assignment

$$(D, D') \mapsto [D, D'] + \varphi(D, D'), \qquad D, D' \in \overline{G},$$

defines a new Lie algebra structure on the vector space underlying $\overline{G}$. Denote by $\overline{L}$ the Lie algebra thus obtained. Since φ is homogeneous of degree 2, $\overline{L}$ is a filtered deformation of $\overline{G}$. Now fix $\mathbf{m} = (2, m_2)$, $m_2 > 1$. Let C be the subspace of $O_2(\mathbf{m})$ spanned by the monomials $x_1^{(r)} x_2^{(s)}$ with $s < 2^{m_2} - 1$, $r = 0, 1, 2, 3$, or $s = 2^{m_2} - 1$, $r = 0, 1$. The subspace $k\partial_1 + C\partial_2$ is a subalgebra of both $\overline{G}$ and $\overline{L}$. Denote by G and L the Lie algebras whose underlying vector space is $k\partial_1 + C\partial_2$ and multiplications are restrictions of those in $\overline{G}$ and $\overline{L}$, respectively. Then L is a filtered deformation of G. The Lie algebra G is semisimple with a unique minimal ideal $A(G) \cong O_1(2) \otimes S$ where $S \cong K_1'(m_2)$. Obviously, $\operatorname{gr}[L, L] \supset [A(G), A(G)] = A(G)$. On the other hand,

$$\left[\partial_2, x_1^{(3)}\partial_2\right] = \partial_1, \qquad \left[x_1 x_2 \partial_2, x_1^{(3)} x_2^{(2^{m_2}-3)}\partial_2\right] = x_2^{(2^{m_2}-1)}\partial_2,$$
$$\left[x_1^{(2)} x_2 \partial_2, x_1^{(3)} x_2^{(2^{m_2}-3)}\partial_2\right] = x_1 x_2^{(2^{m_2}-1)}\partial_2$$

when computed in L. It follows $[L, L] = L$. If I is an ideal of L then $\operatorname{gr} I \supset A(G)$. By Proposition 3.3, $I \supset L^{(\infty)} = L$. Thus L is simple.

THEOREM 6.7. *Let $p = 3$, and let $L = L_{-\alpha} \oplus H \oplus L_\alpha$ be the root space decomposition of a Lie algebra L with respect to a toral rank one Cartan subalgebra H. Then $[L_{-\alpha}, L_\alpha]$ acts triangulably on L. In particular, if L is simple, then H is triangulable.*

Proof. Suppose first that L is simple. Let L_0 be a maximal subalgebra of L containing H and Γ the representation of L_0 in L/L_0. Then $\Gamma(H)$ is a Cartan subalgebra of $\Gamma(L_0)$. Applying Theorem 6.3, we see that $\dim \Gamma(H) = 1$. Hence $[H, H] \subset \ker \Gamma$, and so $\alpha([H, H]) = 0$. Thus $[H, H]$ acts nilpotently on L.

Consider now the general case. Put $H' := [L_{-\alpha}, L_\alpha]$ and $J := L_{-\alpha} + H' + L_\alpha$. We may assume $\alpha(H') \neq 0$. Then H' is a Cartan subalgebra of J and $[J, J] = J$. Let J' be a maximal ideal of J and $\pi : J \to J/J'$ the canonical projection. Then J/J' is a simple algebra and $\pi(H')$ its toral rank 1 Cartan subalgebra. As we have proved already, $\pi([H', H'])$ acts nilpotently on J/J'. Since $J/J' \neq \pi(H')$, the representation of H' in J/J' has nonzero weights. It follows $\alpha([H', H']) = 0$, i.e., $[H', H']$ acts nilpotently on L.

7. EXCEPTIONAL HAMILTONIAN ALGEBRAS OF CHARACTERISTIC 3

In Section 4 we encountered certain hamiltonian Lie algebras of characteristic 3 which contain infinitely many maximal subalgebras of codimension 2. Here we will give a different realization of those algebras which reflects better their intrinsic properties. Finally we determine the orbits of the maximal subalgebras of codimension 2 with respect to the automorphism group.

Given $m > 0$, put $W = W_1(m)$, and $R = O_1(m)$. An (R, W)-module is, by definition, a vector space M endowed with structures of an R-module and a W-module such that $D(fq) = (Df)q + f(Dq)$ for all $f \in R$, $D \in W$, $q \in M$. Suppose that Q is an (R, W)-module satisfying the following additional properties.:

(Q1) $(fD)q = f(Dq)$ for all $f \in R$, $D \in W$, $q \in Q$;

(Q2) Q is free of rank 2 over R;

(Q3) Q is endowed with a nondegenerate skewsymmetric R-bilinear mapping $\langle \cdot , \cdot \rangle : Q \times Q \to R$ which is also W-invariant; in other words, an isomorphism of (R, W)-modules $\bigwedge_R^2 Q \xrightarrow{\sim} R$ is given.

Denote by W_0 the subalgebra of codimension 1 in W, by $\mathfrak{h}$ its normalizer in the universal p-envelope $P(W)$, and by $\mathfrak{m}$ the maximal ideal of R. According to [13, Theorem 2.1; 18, I, Theorem 4.2] the functor $M \mapsto M/\mathfrak{m}M$ is an equivalence between the category of (R, W)-modules and the category of restricted $\mathfrak{h}$-modules. We therefore look at what properties of the restricted $\mathfrak{h}$-module $\overline{Q} = Q/\mathfrak{m}Q$ are imposed by conditions (Q1)–(Q3). The first of these means that $W_0 = \mathfrak{m}W$ annihilates $\overline{Q}$. Thus $\mathfrak{h}$ acts in $\overline{Q}$ via its factor algebra $\mathfrak{h}/P(W_0)$ where $P(W_0)$ is the p-envelope of W_0. The second means just that $\dim \overline{Q} = 2$. The (R, W)-module isomorphism of the third condition corresponds to an isomorphism of $\mathfrak{h}$-modules $\bigwedge^2 \overline{Q} \xrightarrow{\sim} k$, i.e., $\overline{Q}$ is equipped with a nondegenerate skewsymmetric $\mathfrak{h}$-invariant bilinear form. Now $\mathfrak{h}/P(W_0)$ is a restricted algebra generated by a single element ∂^{p^m}, hence is abelian. It follows that $\overline{Q}$ contains a one-dimensional $\mathfrak{h}$-invariant subspace, say $\overline{P}$. If $\lambda : \mathfrak{h} \to k$ is the eigenvalue function of the representation of $\mathfrak{h}$ in $\overline{P}$, then, since the image of $\mathfrak{h}$ in $\mathrm{gl}(\overline{Q})$ is contained in $\mathrm{sp}(\overline{Q})$, the eigenvalue function of the representation of $\mathfrak{h}$ in $\overline{Q}/\overline{P}$ equals $-\lambda$. Three different possibilities can happen. The first one: $\mathfrak{h}$ annihilates $\overline{Q}$; every one-dimensional subspace of $\overline{Q}$ is a trivial $\mathfrak{h}$-module. The second: $\lambda = 0$, however, $\mathfrak{h}$ does not annihilate $\overline{Q}$. In this case $\overline{Q}$ contains just one $\mathfrak{h}$-invariant one-dimensional subspace $\overline{P}$; both $\overline{P}$ and $\overline{Q}/\overline{P}$ are trivial $\mathfrak{h}$-modules. The third: $\lambda \neq 0$. In this case $\overline{Q}$ is a direct sum of weight spaces corresponding to λ and $-\lambda$; any one-dimensional

$\mathfrak{h}$-submodule of $\overline{Q}$ coincides with one of weight spaces. Each $\mathfrak{h}$-submodule $\overline{P} \subset \overline{Q}$ corresponds to an (R, W)-submodule $P \subset Q$. The condition $\dim \overline{P} = 1$ implies that P is free of rank 1 over R, i.e., an invertible (R, W)-module. Recall that an invertible (R, W)-module is said to be trivial if it is isomorphic to R with natural actions of R and W. We see that there are three different types of modules Q:

first type: $Q \cong R \otimes U$, $\dim U = 2$, both R and W act on the first factor of the tensor product;

second type: Q contains exactly one nonzero proper (R, W)-submodule P; both P and Q/P are trivial invertible (R, W)-modules;

third type: $Q \cong P \oplus P^{-1}$ is a direct sum of two nontrivial invertible (R, W)-modules where $P^{-1} := \mathrm{Hom}_R(P, R)$; every nonzero proper (R, W)-submodule of Q coincides with one of those two.

As shown in [20, Sect. 3], with Q there is associated a complex of differential forms $\cdots \to 0 \to Q \to \Omega^1 Q \to 0 \to \cdots$ having nonzero terms in dimensions 0 and 1. Here $\Omega^1 Q := \mathrm{Hom}_R(W, Q)$ and the differential $d : Q \to \Omega^1 Q$ is given by the rule $(dq)(D) = Dq$ for $q \in Q$, $D \in W$. Denote by $H^i Q$ its cohomology groups. In particular, $H^1 Q = \Omega^1 Q / B^1 Q$ where $B^1 Q = dQ$. Let P be an invertible (R, W)-submodule of Q. The short exact sequence $0 \to P \to Q \to Q/P \to 0$ gives rise to a short exact sequence of complexes of differential forms, hence to a long exact sequence of cohomology groups [20, Proposition 3.4]

$$0 \to H^0 P \to H^0 Q \to H^0(Q/P) \to H^1 P \to H^1 Q \to H^1(Q/P) \to 0.$$

According to [18, I, Theorem 7.3], $H^0 P$ and $H^1 P$ are one-dimensional if P is trivial and zero otherwise. The same result applies to Q/P. If Q is either of first or third type then the sequence of modules splits, whence $H^1 Q \cong H^1 P \oplus H^1(Q/P)$. We deduce $\dim H^1 Q = 2$ for the first type and $\dim H^1 Q = 0$ for the third. Suppose that Q is of the second type. Note that $H^0 Q$ is identified with the subspace of Q consisting of elements annihilated by W. If $q \in H^0 Q$ then Rq is an (R, W)-submodule of Q. It follows $Rq \subset P$, i.e., $q \in P$. Thus the mapping $H^0 P \to H^0 Q$ is bijective. Since $\dim H^0(Q/P) = 1 = \dim H^1 P$, the mapping $H^0(Q/P) \to H^1 P$ is bijective as well. Hence $H^1 Q \cong H^1(Q/P)$, We conclude $\dim H^1 Q = 1$ for the second type.

Put $\Omega^1 := \Omega^1 R = \mathrm{Hom}_R(W, R)$. Then $\Omega^1 Q \cong \Omega^1 \otimes_R Q$. Since $\mathrm{rk}_R W = \mathrm{rk}_R \Omega^1 = 1$, there is an (R, W)-module isomorphism $\mu : \Omega^1 \otimes_R \Omega^1 \xrightarrow{\sim} W$ [19, Sect. 1] determined uniquely up to a scalar multiple. We fix μ and put

$$\mathfrak{D}_{\theta, \theta'} := \langle q, q' \rangle \mu(\eta \otimes \eta') \qquad \text{for } \theta = \eta \otimes q, \theta' = \eta' \otimes q',$$

where $\eta, \eta' \in \Omega^1, q, q' \in Q$. By linearity define $\mathfrak{D}_{\theta, \theta'} \in W$ for arbitrary $\theta, \theta' \in \Omega^1 Q$. The assignment $(\theta, \theta') \to \mathfrak{D}_{\theta, \theta'}$ gives a nondegenerate skewsymmetric R-bilinear mapping $\Omega^1 Q \times \Omega^1 Q \to W$. The expression $J(\theta, \theta', \theta'') := \theta(\mathfrak{D}_{\theta', \theta''}) + \theta'(\mathfrak{D}_{\theta'', \theta}) + \theta''(\mathfrak{D}_{\theta, \theta'})$, $\theta, \theta', \theta'' \in \Omega^1 Q$, is skewsymmetric and R-linear in its arguments. Since $\mathrm{rk}_R \Omega^1 Q = 2$, it must be identically zero. Now $D\theta = d(D \lrcorner \theta)$ for $D \in W, \theta \in \Omega^1 Q$, where $D \lrcorner \theta := \theta(D)$ is the inner product. We conclude, applying d to $J(\theta, \theta', \theta'')$, that $\mathfrak{D}_{\theta, \theta'} \theta'' + \mathfrak{D}_{\theta', \theta''} \theta + \mathfrak{D}_{\theta'', \theta} \theta' = 0$. We can now introduce certain Lie algebras

$$X(Q) := W \oplus \Omega^1 Q \qquad \text{and} \qquad X'(Q) := W \oplus B^1 Q.$$

The Lie algebra W is identified with a subalgebra of $X(Q)$ and its action on $\Omega^1 Q$ is the natural one. We put $[\theta, \theta'] = \mathfrak{D}_{\theta, \theta'}$ for $\theta, \theta' \in \Omega^1 Q$. Thus $X(Q)$ has a $\mathbb{Z}_2$-grading. The Jacobi identity in $X(Q)$ follows from the W-invariance of the multiplication and the identity for $\theta, \theta', \theta''$ established above. Obviously $[X(Q), X(Q)] \subset X'(Q)$, so $X'(Q)$ is an ideal of $X(Q)$.

We claim that $X'(Q)$ is a simple algebra. Consider the exact sequence $0 \to \Omega^1 P \to \Omega^1 Q \to \Omega^1(Q/P) \to 0$ where $P, Q/P$ are invertible (R, W)-modules. According to the theory of height one representations of W [10], $\Omega^1 P$ contains a single irreducible W-submodule $B^1 P$ and W acts trivially in the factor module $H^1 P$. The same applies to $\Omega^1(Q/P)$. None of the compositional factors of $\Omega^1 Q$ regarded as a W-module is isomorphic to the adjoint W-module. Hence, if I is an ideal of $X'(Q)$, either $I \supset W$ or $I \cap B^1 Q \neq 0$. Note that the multiplication $\Omega^1 Q \times \Omega^1 Q \to W$ is a nondegenerate R-bilinear mapping. If $\theta \in \Omega^1 Q$ is such that $[\theta, B^1 Q] = 0$ then $[\theta, \Omega^1 Q] = 0$ since $\Omega^1 Q = R \cdot B^1 Q$, and it follows $\theta = 0$. Thus if $I \cap B^1 Q \neq 0$, then $I \cap W \neq 0$ as well, hence $I \supset W$. The subspace $Q' \subset Q$ spanned by all elements Dq with $D \in W$, $q \in Q$ is an (R, W)-submodule. Hence Q/Q' is an (R, W)-module annihilated by W. It follows $Q/Q' = 0$, i.e., $Q' = Q$. Since $D \cdot dq = d(Dq)$ for $D \in W, q \in Q$, we deduce $I \supset dQ' = dQ$ as well.

THEOREM 7.1. *The algebra $X(Q)$ is isomorphic to a hamiltonian Lie algebra, $H_2''(\mathbf{m}, \omega) \subset X(Q) \subset H_2(\mathbf{m}, \omega)$, where $\mathbf{m} = (m, 1)$ and ω is one of $dx_1 \wedge dx_2, (1 + x_1^{(p^m - 1)} x_2^{(2)})\, dx_1 \wedge dx_2$ or $\exp(x_1)\, dx_1 \wedge dx_2$ depending on whether Q is of the first, second, or third type. A Lie algebra L satisfies an inclusion $X'(Q) \subset L \subset X(Q)$ for suitable m and Q if and only if L possesses an exhaustive separating standard filtration such that the associated graded algebra G' satisfies conditions* (1)–(3) *of Proposition* 4.4.

Proof. Suppose L has a filtration with the required properties. By Proposition 4.6, $H_2''(\mathbf{m}, \omega) \subset L \subset H_2(\mathbf{m}, \omega)$ where $\mathbf{m} = (m, 1)$. Moreover, $\mathrm{gr}\, L \subset X(m)$ where $\mathrm{gr}\, L$ is the graded algebra associated with the filtra-

tion of L induced by the standard filtration of $H_2(\mathbf{m}, \omega)$. According to the classification of volume forms [27, 33] or hamiltonian forms [17], which is the same under present settings, there are four possibilities for ω: the three given in the statement of the theorem and the fourth one $\omega = \exp(x_2)\,dx_1 \wedge dx_2$. In the last case $x_2^{(2)}\partial_1 \in H_2''(\mathbf{m}, \omega) = H_2(\mathbf{m}, \omega)$, how-
ever; whence $x_2^{(2)}\partial_1 \in \operatorname{gr} H_2''(\mathbf{m}, \omega)$, i.e., $\operatorname{gr} H_2''(\mathbf{m}, \omega) \not\subset X(m)$. Thus the fourth possibility should be excluded.

Now we shall construct a filtration in $X(Q)$. Let $\mathfrak{m}^{(j)}$, $j \geq 0$, be the canonical filtration in R [9]. Put

$$X(Q)_i = \begin{cases} \mathfrak{m}^{(j+1)}W \oplus \mathfrak{m}^{(j+1)}\Omega^1 Q & \text{if } i = 2j \text{ is even,} \\ \mathfrak{m}^{(j+1)}W \oplus \mathfrak{m}^{(j)}\Omega^1 Q & \text{if } i = 2j-1 \text{ is odd.} \end{cases}$$

Denote by G' the associated graded algebra. We have

$$G_i' \cong \mathfrak{m}^{(j+1)}W/\mathfrak{m}^{(j+2)}W = \operatorname{gr}_j W \cong \operatorname{gr}_{j+1} R \otimes \overline{W} \qquad \text{for } i = 2j,$$

$$G_i' \cong \mathfrak{m}^{(j)}\Omega^1 Q/\mathfrak{m}^{(j+1)}\Omega^1 Q \cong \operatorname{gr}_j R \otimes \overline{\Omega}^1 \otimes \overline{Q} \qquad \text{for } i = 2j-1.$$

where $\overline{W} := W/\mathfrak{m}W$ and $\overline{\Omega}^1 := \Omega^1/\mathfrak{m}\,\Omega^1$. Thus $\dim G_{2j}' = 1$, $\dim G_{2j+1}' = 2$ for $-1 \leq j < 3^m - 1$. Furthermore, condition (2) of Proposition 4.4 is fulfilled since W_0, hence also $G_0' \cong \operatorname{gr}_0 W$, annihilates $\overline{Q}$. The multiplication mapping $G_{2j-1}' \times G_{2l-1}' \to G_{2j+2l-2}'$ is composed of the multiplication $\operatorname{gr}_j R \times \operatorname{gr}_l R \to \operatorname{gr}_{j+l} R$ in $\operatorname{gr} R$, the isomorphism $\overline{\Omega}^1 \otimes \overline{\Omega}^1 \to \overline{W}$ induced by μ, and the mapping $\overline{Q} \times \overline{Q} \to k$ corresponding to the bilinear form on Q. One sees that (3) is fulfilled as well.

By the above $H_2''(\mathbf{m}, \omega) \subset X(Q) \subset H_2(\mathbf{m}, \omega)$ for some $\mathbf{m}$ and ω one of the three types. Then $X'(Q) = H_2''(\mathbf{m}, \omega)$, both being simple ideals of $X(Q)$. Note that $\dim X(Q) = 3 \dim R = 3^{m+1}$. Since $X(Q)/X'(Q) \cong H^1 Q$, we have $\dim X'(Q) = 3^{m+1} - 2$, $3^{m+1} - 1$, or 3^{m+1} depending on the type of Q. Comparing this with the dimension of $H_2''(\mathbf{m}, \omega)$, we conclude that $\mathbf{m} = (m, 1)$ and ω is such as stated for each type of Q.

It remains only to prove that $L \subset X(Q)$ provided that to L and $X(Q)$ there correspond the same m and ω. Note that $\operatorname{gr} X(Q) = X(m)$ since both algebras have dimension 3^{m+1} and $\operatorname{gr} X(Q) \subset X(m)$. Note also that $X(m)$ and $H_2(\mathbf{m})$ have the same homogeneous components except in degree 1. Hence $H_2(\mathbf{m}, \omega)_2 \subset X(Q)$. Since $\operatorname{gr}_1 L \subset X(m)$, it follows $L_1 \subset X(Q)$. Finally, $L = H_2''(\mathbf{m}, \omega) + L_1 \subset X(Q)$ because $\operatorname{gr}_i L = \operatorname{gr}_i H_2''(\mathbf{m}, \omega)$ for $i = -1, 0$.

For a Lie algebra L we introduce an invariant subset

$$\mathfrak{N}(L) = \{u \in L \mid (\operatorname{ad} u)(\operatorname{ad} v) \text{ is nilpotent for every } v \in L\}.$$

LEMMA 7.2. *Suppose that* $L = L'_{-2} \supset L'_{-1} \supset L'_0 \supset \dots$ *is an exhaustive separating standard filtration such that the associated graded algebra* G' *satisfies conditions* (1)–(3) *of Proposition* 4.4. *Suppose* $G'_4 \neq 0$. *Then* L'_0 *is the unique subalgebra of codimension* 3 *in* L *containing* $\mathfrak{N}(L)$. *A subalgebra* M *of codimension* 2 *in* L *contains* $\mathfrak{N}(L)$ *if and only if* $L'_{-1} \supset M \supset L'_0$.

The proof of this lemma is quite similar to that of [19, Proposition 4.5] or [18, II, Proposition 5.5]. We therefore omit easy verifications.

PROPOSITION 7.3. *Suppose that* $m > 1$ *and* $L = X(Q)$ *or* $X'(Q)$. *The algebraic variety* $\mathscr{M}$ *of all subalgebras in* L *having codimension* 2 *and containing* $\mathfrak{N}(L)$ *is isomorphic to the projective line* $\mathbb{P}^1$. *Denote by* $\mathfrak{h}'$ *the normalizer of* L'_0 *in the universal p-envelope* $P(L)$. *The* $\mathfrak{h}'$*-invariant subalgebras in* $\mathscr{M}$ *constitute one orbit with respect to the automorphism group* Aut L. *This orbit coincides with* $\mathscr{M}$ *if* Q *is of the first type, contains a single* (*hence invariant*) *subalgebra for the second type, and contains exactly two subalgebras for the third. The subalgebras which are not* $\mathfrak{h}'$*-invariant constitute the other orbit if* Q *is of the second or third type.*

Proof. According to Lemma 7.2 and Proposition 4.6, $M \in \mathscr{M}$ if and only if M is a subspace of L such that $L'_{-1} \supset M \supset L'_0$ and $\dim M/L'_0 = 1$. Hence the elements of $\mathscr{M}$ are in a one-to-one correspondence with the one-dimensional subspaces of the two-dimensional space $G'_{-1} := L'_{-1}/L'_0$. This gives an isomorphism $\mathscr{M} \cong \mathbb{P}^1$. The group Aut L operates naturally on $\mathscr{M}$. The intersection of all subalgebras $M \in \mathscr{M}$ is L'_0, while their sum is L'_{-1}. Hence L'_0, L'_{-1} are stable under Aut L.

The $\mathbb{Z}_2$-grading of L induces a $\mathbb{Z}_2$-grading of $P(L)$. The subalgebra $\mathfrak{h}'$ is homogeneous with respect to this grading. Denote by $\bar{0}, \bar{1}$ the elements of $\mathbb{Z}_2$. The $\mathfrak{h}'$-module G'_{-1} has zero component of degree $\bar{0}$, whence $\mathfrak{h}'_{\bar{1}}$ annihilates G'_{-1}. Now $\mathfrak{h}'_{\bar{0}} \subset P(W)$. Obviously, $W_0 = L'_0 \cap W$ is stable under $\mathfrak{h}'_{\bar{0}}$, i.e., $\mathfrak{h}'_{\bar{0}} \subset \mathfrak{h}$. Since $\mathfrak{m}$ is stable under $\mathfrak{h}$, so is $L'_0 = (W_0 + \mathfrak{m}\Omega^1 Q) \cap L$ as well, whence $\mathfrak{h}'_{\bar{0}} = \mathfrak{h}$. Note that the canonical isomorphism $G'_{-1} \cong \overline{\Omega}^1 \otimes \overline{Q}$ is $\mathfrak{h}$-invariant. Thus the $\mathfrak{h}'$-invariant subalgebras in $\mathscr{M}$ are in a one-to-one correspondence with the $\mathfrak{h}$-invariant one-dimensional subspaces of $\overline{Q}$. The totality of the latter has been described already.

Denote by $\mathscr{G}$ the set of all pairs (φ, ψ) where φ is an admissible automorphism of R, so that there are the induced automorphisms $\varphi_* : W \to W$ and $\varphi_* : \Omega^1 \to \Omega^1$, and ψ a k-linear invertible transformation of Q such that $\psi(fq) = \varphi(f)\psi(q)$, $\psi(Dq) = \varphi_*(D)\psi(q)$, and $\langle \psi(q), \psi(q') \rangle = \varphi(\langle q, q' \rangle)$ for all $f \in R, D \in W, q, q' \in Q$. We make $\mathscr{G}$ into a group by using componentwise composition. With each $(\varphi, \psi) \in \mathscr{G}$ we associate an automorphism of $X(Q)$ which acts as φ_* on W and as $a(\varphi)\varphi_* \otimes \psi$ on $\Omega^1 \otimes_R Q$, where $a(\varphi)$ is a certain nonzero scalar given by the condition $a(\varphi)\mu \circ (\varphi_* \otimes \varphi_*) = \varphi_* \circ \mu$. Thus $\mathscr{G}$ is a group of automorphisms of

$X(Q)$ preserving $\mathbb{Z}_2$-grading. The isomorphism $G'_{-1} \cong \overline{\Omega}^1 \otimes \overline{Q}$ is obviously $\mathscr{G}$-equivariant. We will show that any two one-dimensional subspaces of $\overline{Q}$ are conjugate with respect to $\mathscr{G}$ provided that they are both either $\mathfrak{h}$-invariant or not $\mathfrak{h}$-invariant. Given an admissible automorphism φ of R denote by ${}^{\varphi}Q$ the (R, W)-module whose underlying vector space is that of Q, while the actions of R and W are obtained by composing the original ones with φ^{-1} and φ_*^{-1}. Endow ${}^{\varphi}Q$ with a skewsymmetric R-bilinear form ${}^{\varphi}\langle q, q' \rangle := \varphi(\langle q, q' \rangle)$, $q, q' \in Q$. Then ${}^{\varphi}Q$ satisfies (Q1)–(Q3). The condition that $(\varphi, \psi) \in \mathscr{G}$ means precisely that ψ is an isomorphism of (R, W)-modules ${}^{\varphi}Q \to Q$ preserving bilinear forms. Those ψ are in a one-to-one correspondence with the $\mathfrak{h}$-module isomorphisms $\overline{\psi} : {}^{\varphi}\overline{Q} \to \overline{Q}$ preserving bilinear forms. Here ${}^{\varphi}\overline{Q}$ is the $\mathfrak{h}$-module whose underlying vector space is that of $\overline{Q}$, while the action of $\mathfrak{h}$ is twisted by φ_*. The bilinear form associated with ${}^{\varphi}\overline{Q}$ is unchanged because φ induces identity transformation of $R/\mathfrak{m} \cong k$. Examine first the pairs (φ, ψ) with $\varphi = \mathrm{id}$, so that ${}^{\varphi}\overline{Q} = \overline{Q}$. If Q is of the first type then $\overline{\psi}$ is an arbitrary symplectic transformation. The group $\mathrm{Sp}(\overline{Q})$ permutes transitively the one-dimensional subspaces. If Q is of the second type then $\overline{\psi}$ can be taken arbitrary subject to the conditions that the unique one-dimensional $\mathfrak{h}$-submodule $\overline{P} \subset \overline{Q}$ is stable under $\overline{\psi}$ and $\overline{\psi}$ induces identity transformations in both $\overline{P}$ and $\overline{Q}/\overline{P}$. The group of those $\overline{\psi}$ permutes transitively the one-dimensional subspaces distinct from $\overline{P}$. Let Q be of the third type. Then $\overline{Q} = \overline{P} \oplus \overline{P}'$ is a sum of weight spaces with respect to $\mathfrak{h}$ corresponding to eigenvalue functions $\lambda, -\lambda$. Here $\overline{\psi}$ acts as a scalar multiplication on $\overline{P}, \overline{P}'$ with eigenvalues β, β^{-1}, where nonzero β can be taken arbitrary. The group of those $\overline{\psi}$ permutes transitively the one-dimensional subspaces distinct from $\overline{P}, \overline{P}'$. To show that $\overline{P}, \overline{P}'$ are conjugate with respect to $\mathscr{G}$ we need to employ nonidentity φ. Let $\varphi(x) = -x$. Then $\varphi_*(\partial) = -\partial$, hence $\varphi_*(\partial^{p^m}) = -\partial^{p^m}$. It follows that in the decomposition ${}^{\varphi}\overline{Q} = {}^{\varphi}\overline{P} \oplus {}^{\varphi}\overline{P}'$ the two subspaces are weight spaces corresponding to eigenvalue functions $-\lambda$ and λ, respectively. We can find $\overline{\psi} \in \mathrm{Sp}(\overline{Q})$ such that $\overline{\psi}(\overline{P}) = \overline{P}'$, $\overline{\psi}(\overline{P}') = \overline{P}$. Then (φ, ψ) interchanges $\overline{P}$ and $\overline{P}'$.

ACKNOWLEDGMENTS

The author gratefully acknowledges financial support of the Alexander von Humboldt Foundation and the hospitality of the Mathematical Department of Hamburg University during the time when the work was conducted. Numerous conversations with H. Strade and the opportunity to use first chapters of his manuscript on classification of simple Lie algebras stimulated a quick progress of research.

REFERENCES

1. G. M. Benkart and J. M. Osborn, Toral rank one Lie algebras, *J. Algebra* **115** (1988), 238–250.
2. R. Block, Determination of the differentiably simple rings with a minimal ideal, *Ann. of Math.* **90** (1969), 433–459.
3. R. E. Block and R. L. Wilson, Classification of the restricted simple Lie algebras, *J. Algebra* **114** (1988), 115–259.
4. A. S. Dzhumadil'daev, Simple Lie algebras with a subalgebra of codimension 1, *Uspekhi Mat. Nauk* **40**, No. 1 (1985), 193–94. [In Russian]
5. V. Guillemin and S. Sternberg, An algebraic model of transitive differential geometry, *Bull. Amer. Math. Soc.* **70** (1964), 16–47.
6. I. Hayashi, Embedding and existence theorems of infinite Lie algebras, *J. Math. Soc. Japan* **22** (1970), 1–14.
7. N. Jacobson, A note on three-dimensional simple Lie algebras, *J. Math. Mech.* **7** (1958), 823–831.
8. I. Kaplansky, Lie algebras of characteristic p, *Trans. Amer. Math. Soc.* **89** (1958), 149–183.
9. A. I. Kostrikin and I. R. Shafarevich, Graded Lie algebras of finite characteristic, *Izv. Akad. Nauk SSSR Ser. Math.* **33** (1969), 251–322. [In Russian]
10. Y. S. Kryluk, ''Algebras of Cartan Type: Representations and Prologations,'' Dissertation, Moscow State University, Moscow, 1978. [In Russian]
11. M. I. Kuznetsov, Modular simple Lie algebras with a solvable maximal subalgebra, *Mat. Sb.* **101** (1976), 77–86. [In Russian]
12. M. I. Kuznetsov, The classification of the simple graded Lie algebras with nonsemisimple component L_0, *Mat. Sb.* **180** (1989), 147–158. [In Russian]
13. M. I. Kuznetsov, Truncated induced modules over transitive Lie algebras of characteristic p, *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), 557–589. [In Russian]
14. A. A. Premet, Regular Cartan subalgebras and nilpotent elements in restricted Lie algebras, *Mat. Sb.* **180** (1989), 542–557. [In Russian]
15. A. A. Premet, Generalization of Wilson's theorem on Cartan subalgebras of simple Lie algebras, *J. Algebra* **167** (1994), 641–703.
16. D. S. Rim, Deformations of transitive Lie algebras, *Ann. of Math.* **83** (1966), 339–357.
17. S. M. Skryabin, Classification of hamiltonian forms over divided power algebras, *Mat. Sb.* **181** (1990), 114–133 [In Russian]; English translation, *Math. USSR-Sb.* **69** (1991), 121–141.
18. S. M. Skryabin, Modular Lie algebras of Cartan type over algebraically non-closed fields, I. *Comm. Algebra* **19** (1991), 1629–1741; II, *Comm. Algebra* **23** (1995), 1403–1453.
19. S. M. Skryabin, New series of simple Lie algebras of characteristic 3, *Mat. Sb.* **183** (1992), 3–22 [In Russian]; English translation, *Russian Acad. Sci. Sb. Math.* **76** (1993), 389–406.
20. S. M. Skryabin, An algebraic approach to the Lie algebras of Cartan type, *Comm. Algebra* **21** (1993), 1229–1336.
21. H. Strade, Darstellungen auflösbarer Lie p-algebren, *Math. Ann.* **232** (1978), 15–32.
22. H. Strade, The absolute toral rank of a Lie algebra, *in* Lecture Notes in Math., Vol. 1373, pp. 1–28, Springer-Verlag, New York/Berlin, 1989.
23. H. Strade, The classification of the simple modular Lie algebras. I. Determination of the two-sections, *Ann. of Math.* **130** (1989), 643–677.
24. H. Strade, The classification of the simple modular Lie algebras. III. Solution of the classical case, *Ann. of Math.* **133** (1991), 577–604.
25. H. Strade, The classification of the simple Lie algebras over fields with positive characteristic, ''Hamburger Beiträge zur Mathematik,'' Heft 31, Hamburg, 1997.

26. H. Strade and R. Farnsteiner, "Modular Lie Algebras and Their Representations," Marcel Dekker Textbooks and Monographs, Vol. 116, Dekker, New York, 1988.
27. S. A. Tyurin, Classification of deformations of a special Lie algebra of Cartan type, *Mat. Zametki* **24** (1978), 847–857. [In Russian]
28. B. Weisfeiler, On the structure of the minimal ideal of some graded algebras in characteristic $p > 0$, *J. Algebra* **53** (1978), 344–361.
29. B. Weisfeiler, On subalgebras of simple Lie algebras of characteristic $p > 0$, *Trans. Amer. Math. Soc.* **286** (1984), 471–503.
30. R. L. Wilson, A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic, *J. Algebra* **40** (1976), 418–465.
31. R. L. Wilson, Cartan subalgebras of simple Lie algebras, *Trans. Amer. Math. Soc.* **234** (1977), 436–446; Correction, *Trans. Amer. Math. Soc.* **305** (1988), 851–855.
32. R. L. Wilson, Simple Lie algebras of toral rank one, *Trans. Amer. Math. Soc.* **236** (1978), 287–295.
33. R. L. Wilson, Simple Lie algebras of type S, *J. Algebra* **62** (1980), 292–298.